On Moufang loops and related groups

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(based on joint research with A. N. Grishkov)

Moufang loops

A *groupoid* (or *magma*) $(M, \cdot)$ is just a set M with a binary operation '$\cdot$'

A groupoid M is a *quasigroup* if

- $L_m : x \mapsto m \cdot x, \quad R_m : x \mapsto x \cdot m$ are bijections of M $\forall\, m \in M$,

A quasigroup M is a *loop* if

- $\exists\, e \in M \;:\; e \cdot x = x \cdot e = x \;\; \forall\, x \in M.$ (the *identity* element)

Loops are "nonassociative groups".

Convention: $xy = x \cdot y, \quad xy \cdot z = (x \cdot y) \cdot z, \quad x \cdot yz = x \cdot (y \cdot z),$ etc.

Definition. A loop M is *Moufang* if it satisfies one (hence, all) of the following *Moufang identities*:

- $(xy \cdot x)z = x(y \cdot xz)$ (*left*)
- $x(y \cdot zy) = (xy \cdot z)y$ (*right*)
- $xy \cdot zx = (x \cdot yz)x$ (*left central*)
- $xy \cdot zx = x(yz \cdot x)$ (*right central*)

Moufang loops are *diassociative*: $\forall x, y \in M$, $\langle x, y \rangle$ is a group.

In particular, $x.yx = xy.x = xyx$

- [R. Moufang, 1935] For $x, y, z \in M$, $(x, y, z) = 1 \;\Rightarrow \langle x, y, z\rangle$ is assoc.

The associator $a = (x, y, z)$ is defined by $xy \cdot z = (x \cdot yz)a$.

- [A. Drápal, 2010] A simplified proof of Moufang's theorem.

Groups with triality

Definition. A group G is *with triality* if there are $\rho, \sigma \in \mathrm{Aut}(G)$ satisfying

$$\rho^3 = \sigma^2 = (\rho\sigma)^2 = \varepsilon,$$

$$(x^{-1}x^{\sigma})(x^{-1}x^{\sigma})^{\rho}(x^{-1}x^{\sigma})^{\rho^2} = 1 \qquad \forall x \in G.$$

G is a group with triality $S = \langle \sigma, \rho\rangle$ (an image of Sym_3).

Operation '$\star$' on $M = \{x^{-1}x^{\sigma} \mid x \in G\}$: for $m, n \in M$, set

$$m \star n = m^{-\rho} n m^{-\rho^2}$$

- $(M, \star) = \mathcal{M}(G)$ is a Moufang loop.
- Every ML has the form $\mathcal{M}(G)$ for some group w/triality G.
- $H \leqslant_S G \quad \Rightarrow \quad \mathcal{M}(H) \leqslant \mathcal{M}(G)$.
- $L \leqslant \mathcal{M}(G) \quad \Rightarrow \quad \exists\, H \leqslant_S G : \; \mathcal{M}(H) = L$.

TRIALITY ACTION OF G ON $\mathcal{M}(G)$

G is a group w/triality $S = \langle \rho, \sigma \rangle$, $M = \mathcal{M}(G)$, $g \in G$.

- $\chi(g) : m \mapsto g^{-1} m g^{\sigma}$ is a permutation of M.
- $\chi : G \to \mathrm{Sym}(M)$ is a homomorphism.

For $m \in M$, $\chi(m^{\rho}) = L_m$, $\chi(m^{\rho^2}) = R_m$, $\chi(m) = P_m = L_{m^{-1}} R_{m^{-1}}$.

- $\mathrm{Mlt}(M) = \langle L_m, R_m \mid m \in M \rangle$ lies in $\mathrm{Im}(\chi)$.

Since $\rho : M \to M^{\rho} \to M^{\rho^2} \to M$, $\sigma : M^{\rho} \leftrightarrow M^{\rho^2}$, $M^{\sigma} = M$,

we have *the triality* (no homomorphism $S \to \mathrm{Aut}(\mathrm{Mlt}(G))$ in general)

$$P_m \overset{\rho}{\longmapsto} L_m \overset{\rho}{\longmapsto} R_m \overset{\rho}{\longmapsto} P_m$$

$$L_m \overset{\sigma}{\longmapsto} R_m^{-1}, \quad R_m \overset{\sigma}{\longmapsto} L_m^{-1}, \quad P_m \overset{\sigma}{\longmapsto} P_m^{-1}.$$

For M with $\mathrm{Nuc}(M) = 1$ and $G = \mathrm{Mlt}(M)$, this dates back to [G. Glauberman, 1968].

$\mathrm{Nuc}(M) = \{m \in M \mid (m, M, M) = (M, m, M) = (M, M, m) = 1\} \trianglelefteq M$

Groups with triality generalise the classical triality on $\mathrm{Mlt}(M)$ for a general Moufang loop M.

Multiplication formulas

$M = LN$, where L, N are subloops of M.

$l_1 n_1 \cdot l_2 n_2 = ln$, where $l_1, l_2, l \in L$, $n_1, n_2, n \in N$.

Example. $l_1 n_1 \cdot l_2 n_2 = l_1 l_2 \cdot n_1^{l_2} n_2$ if M is a group and $N \trianglelefteq M$.

Many new (Moufang) loops are defined in a similar way.

The loop $\mathcal{M}(G)$ has a generic multiplication formula.

- Let G be a group w/triality $S = \langle \rho, \sigma \rangle$ and $l, k, n, m \in \mathcal{M}(G)$ then

$$(l \star n) \star (k \star m) = (l \star k) \star x, \qquad (*)$$

where

$$x = n^{-\rho k^{-\rho} l^{\rho^2}} m^{[k^{\rho^2}, l^{-\rho}]} n^{-\rho^2 k^{\rho^2} l^{-\rho}} \in \mathcal{M}(G).$$

- Let K and V be S-subgroups of a group G w/triality. Suppose $V \trianglelefteq G$ and $G = KV$. Then $(*)$ is a multiplication formula in $M = \mathcal{M}(G)$ w.r.t. the decomposition $M = \mathcal{M}(K) \star \mathcal{M}(V)$, where $\mathcal{M}(V) \trianglelefteq M$ and $l, k \in \mathcal{M}(K)$, $n, m \in \mathcal{M}(V)$.

- If $M = LN$ with $N \trianglelefteq M$, there is a group w/triality G as above such that $\mathcal{M}(G) = M$, $\mathcal{M}(K) = L$, $\mathcal{M}(V) = N$.

Normal abelian subloops

M is a Moufang loop, $x, y \in M$.

Define operators $T_x = R_x L_x^{-1}$, $L_{x,y} = L_x L_y L_{yx}^{-1}$, $D_{x,y} = L_x R_y L_{xy}^{-1}$

- If $N \trianglelefteq M$ then N is invariant under T_x, $L_{x,y}$, $D_{x,y}$.

Example. $ln \cdot km = lk \cdot (nT_k + m)$ if $M = LN$ is a group, where $N \trianglelefteq M$, N is abelian, $l, k \in L$, $n, m \in N$.

- As above, let $G = KV$ with K and V being S-subgroups and $V \trianglelefteq G$ abelian. Then the multiplication formula in $M = \mathcal{M}(G)$ has the *inner* form

$$(l \star n) \star (k \star m) = (l \star k) \star x, \qquad x = nD_{l,k} + mL_{k,l}, \qquad (*)$$

where $l, k \in \mathcal{M}(K)$, $n, m \in \mathcal{M}(V) \trianglelefteq \mathcal{M}(G)$.

Problem. Does $M = LN$ with N normal abelian admit mult. formula $(*)$?

- A necessary condition: $(M, N, N) = 1$. ($\Rightarrow$ $D_{l,k}$, $L_{k,l}$ are linear on N)
- A series of *abelian-by-cyclic* MLs (i. e. with N abelian, L cyclic) of orders 3.2^6, 7.2^9, 3.5^6, 3.7^3, ... with $(M, N, N) \neq 1$.

Wreathlike triality groups

H is a group. $T = H \times H \times H$ is naturally a group w/triality $S = \langle \rho, \sigma \rangle$.

- $\mathcal{M}(T) \cong H$

R is a comm. assoc. ring, V is an RH-module free of R-rank n.

- $W = V \# V \# V$ is an RT-module with natural S-action.

Problem. When is $T \leftthreetimes W$ a group with triality S?

- $T \leftthreetimes W$ has triality S if and only if $n \leqslant 2$.
- If $n = 1$ then $\mathcal{M}(T \leftthreetimes W) \cong H$. The case $n = 2$ is of main interest.
- Let $G \leqslant \mathrm{GL}_2(R)$ and let V the free R-module of rank 2 with the natural action "$\circ$" of G. Denote by $G \leftthreetimes_M V$ the set of pairs (g, u) for $g \in G$, $u \in V$. Then

$$(g, u) \cdot (h, w) = (gh, u \circ (\det h) g h^{-2} g^{-1} + w \circ [h^{-1}, g^{-1}])$$

defines a Moufang loop structure on $G \leftthreetimes_M V$. This Moufang loop is isomorphic to $\mathcal{M}(T \leftthreetimes W)$, where $W = V \# V \# V$ is as above.

- $D_{g,h} \;\leftrightarrow\; (\det h) g h^{-2} g^{-1}, \quad L_{h,g} \;\leftrightarrow\; [h^{-1}, g^{-1}]$

We call $G \leftthreetimes_M V$ the (outer) *Moufang semidirect product* of G and V.

- $G \leftthreetimes_M V$ is nonassociative iff G is nonabelian.
- $\mathrm{Sc}(G) \trianglelefteq G \leftthreetimes_M V$, where $\mathrm{Sc}(G) = \{\text{scalars of } G\}$.

$\Rightarrow$ Moufang loops $\overline{G} \leftthreetimes_M V$, where $\overline{G} = G/\mathrm{Sc}(G)$.

The structure of $\mathrm{GL}_2(q)$ $\Rightarrow$ existence of *abelian-by-simple* finite MLs:

- $\mathrm{PSL}_2(q) \leftthreetimes_M \mathbb{F}_q^2$, where $q \geqslant 4$ is a prime power;
- $A_5 \leftthreetimes_M \mathbb{F}_p^2$, where $p \equiv \pm 1 \pmod{10}$ is prime;
- $A_5 \leftthreetimes_M \mathbb{F}_{p^2}^2$, where $p \equiv \pm 3 \pmod{10}$ is prime.

Embedding into $\mathbb{O}(R)$

R is an associative commutative ring with 1.

The Cayley algebra $\mathbb{O}(R)$ is the set of *Zorn matrices*

$$\begin{pmatrix} a & \mathbf{v} \\ \mathbf{w} & b \end{pmatrix}, \quad a, b \in R, \quad \mathbf{v}, \mathbf{w} \in R^3,$$

viewed as a free R-module (of rank 8) with multiplication

$$\begin{pmatrix} a_1 & \mathbf{v}_1 \\ \mathbf{w}_1 & b_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & \mathbf{v}_2 \\ \mathbf{w}_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + \mathbf{v}_1 \cdot \mathbf{w}_2 & a_1 \mathbf{v}_2 + b_2 \mathbf{v}_1 \\ a_2 \mathbf{w}_1 + b_1 \mathbf{w}_2 & \mathbf{w}_1 \cdot \mathbf{v}_2 + b_1 b_2 \end{pmatrix} + \begin{pmatrix} 0 & -\mathbf{w}_1 \times \mathbf{w}_2 \\ \mathbf{v}_1 \times \mathbf{v}_2 & 0 \end{pmatrix},$$

$\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3)$, $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$,

$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2,\ v_3 w_1 - v_1 w_3,\ v_1 w_2 - v_2 w_1) \in R^3$

- $\mathbb{O}(R)$ is an alternative algebra. ($xx \cdot y = x \cdot xy$, $y \cdot xx = yx \cdot x$)
- $\mathbb{O}(R)^\times$ is a Moufang loop (invertible elements of $\mathbb{O}(R)$).

The *parabolic subloop* $\mathbb{P}(R) \leqslant \mathbb{O}(R)^\times$ consists of the matrices

$$\begin{pmatrix} a_{11} & (0, a_{12}, r_1) \\ (r_2, a_{21}, 0) & a_{22} \end{pmatrix},$$

where $(a_{ij}) \in \mathrm{GL}_2(R)$ and $r_1, r_2 \in R$.

- $\mathrm{GL}_2(R) \leftthreetimes_M R^2 \cong \mathbb{P}(R)$.

Action of Moufang loops on abelian groups

R is a commut. assoc. ring, A is an alternative R-algebra

$L_x : a \mapsto xa$, $R_x : a \mapsto ax$ belong to $\mathrm{GL}(A)$ for $x \in A^\times$

$$L_{x,y} = L_x L_y L_{yx}^{-1}, \qquad D_{x,y} = L_x R_y L_{xy}^{-1}.$$

- Let M be a subloop of $A^\times$ and U a subgroup of the additive group of A that is invariant under the operators $L_{n,m}$ and $D_{n,m}$ for all $m, n \in M$. Then the set of pairs (m, u) for $m \in M$, $u \in U$ which we denote by $M \leftthreetimes U$ becomes a Moufang loop w.r.t. the operation

$$(m, u) \cdot (n, w) = (mn, uD_{m,n} + wL_{n,m}).$$

We call $M \leftthreetimes U$ the *outer semidirect product* of M and U.

- Natural embeddings $M \to M \leftthreetimes U$ and $U \to M \leftthreetimes U$.
- The "inner" loop action of $L_{n,m}$, $D_{n,m}$ on U in $M \leftthreetimes U$ coincides with the linear action of $L_{n,m}$, $D_{n,m}$ on U in A for all $m, n \in M$.
- $\mathrm{Sc}(M) \trianglelefteq M \leftthreetimes U$, where $\mathrm{Sc}(M) = R1 \cap M$.

$\Rightarrow$ Moufang loops $\overline{M} \leftthreetimes V$, where $\overline{M} = M/\mathrm{Sc}(M)$.

Semidirect products for simple Moufang loops

$\mathbb{O} = \mathbb{O}(\mathbb{F}_q)$ is equipped with a quadratic form ($=$ *norm*).

$\mathrm{SL}(\mathbb{O}) \leqslant \mathbb{O}^\times$ the subloop of elements with norm 1.

$\mathrm{M}(q) = \mathrm{PSL}(\mathbb{O}) = \mathrm{SL}(\mathbb{O})/\,\mathrm{Sc}(\mathrm{SL}(\mathbb{O}))$, the simple Paige–Moufang loop over $\mathbb{F}_q$.

- [M. Liebeck, 1987] Every nonassociative finite simple Moufang loop is isomorphic to some $\mathrm{M}(q)$.

$1^\perp$ is 7-dimensional and $\mathrm{SL}(\mathbb{O})$-invariant.

The following abelian-by-simple finite Moufang loops exist:

- $\mathrm{M}(q) \leftthreetimes \mathbb{F}_q^7$, where q is an odd prime power;
- $\mathrm{M}(q) \leftthreetimes \mathbb{F}_q^6$, where q is a power of 2;
- $\mathrm{M}(2) \leftthreetimes \mathbb{F}_p^7$, where p is an odd prime.

For $\mathbb{O}$ the classical octonions over $\mathbb{R}$, we have the loop

- $\mathrm{PSL}(\mathbb{O}) \leftthreetimes \mathbb{R}^7$.

Linear representations of Moufang loops

Classification (up to equivalence) of short exact sequences of MLs

$$0 \to V \to E \to M \to 1 \qquad (*)$$

with V an abelian group. A *finite* extension E is said to be

- *minimal*, if $N \trianglelefteq E$, $N \subseteq V \quad \Rightarrow \quad N = V$ or $N = 0$;
- *nontrivial*, if E is nonassociative and $\not\cong V \times M$.

Minimal nontrivial extensions for finite simple noncyclic Moufang loops:

- $\mathrm{PSL}_2(q) \leftthreetimes_M \mathbb{F}_q^2$, $q \geqslant 4$ is a prime power;
- $A_5 \leftthreetimes_M \mathbb{F}_p^2$, $p \equiv \pm 1 \pmod{10}$ is prime;
- $A_5 \leftthreetimes_M \mathbb{F}_{p^2}^2$, $p \equiv \pm 3 \pmod{10}$ is prime;
- $\mathrm{M}(q) \leftthreetimes \mathbb{F}_q^7$, q is an odd prime power;
- $\mathrm{M}(q) \leftthreetimes \mathbb{F}_q^6$, q is a power of 2;
- $\mathrm{M}(2) \leftthreetimes \mathbb{F}_p^7$, p is an odd prime;
- $\mathrm{SL}(\mathbb{O}(q))$, q is an odd prime power ($V \cong \mathbb{Z}/2\mathbb{Z}, M = \mathrm{M}(q)$);
- $0 \to \mathbb{Z}/2\mathbb{Z} \to E \to \mathrm{M}(2) \to 1$.

Problem. Are these all such extensions?

Identities in Moufang loops

Groups are (multiplicative) subgroups of associative algebras.

- [I. Shestakov, 2003] Not all MLs are subloops of alternative algebras.

The free (countable) group embeds into $\mathrm{GL}_2(\mathbb{Z})$.

Problem. Does a free ML embed into $\mathbb{O}^\times$, where $\mathbb{O} = \mathbb{O}(\mathbb{Z})$?

Problem. Does a nontrivial identity hold in $\mathbb{O}^\times$?

An identity is *trivial* if it holds in free Moufang loops (or, equivalently, follows from the Moufang identities).

Example. Trivial identities:

- $(x \cdot yz \cdot x)y = xy \cdot z \cdot xy = x(y \cdot zx \cdot y)$; (two-sided Moufang)
- $z^{-1}(zx \cdot y) = xz^{-1} \cdot zy = (x \cdot yz^{-1})z$;
- $z^{-1}(xz \cdot y) = z^{-1}x \cdot zy; \quad xz^{-1} \cdot yz = (x \cdot z^{-1}y)z$;
- $(z^{-1}x \cdot y)z = z^{-1}(x \cdot yz)$;
- $((xy \cdot z)t \cdot y^{-1})x^{-1} = x(y \cdot z(t \cdot y^{-1}x^{-1}))$;

(Computer-aided) search for identities in $\mathbb{O}^\times$ of small degree.

Sketch of the algorithm:

- $\mathcal{W} = \mathcal{W}_{d,n}$, the set of all words of degree d in n variables
- $a_1 \in (\mathbb{O}^\times)^n$, a random n-tuple of elements
- Substituting a_1 into all words in $\mathcal{W}$ we get l_1 distinct values
- $\mathcal{W} = \mathcal{W}_1^{(1)} \cup \mathcal{W}_2^{(1)} \cup \ldots \cup \mathcal{W}_{l_1}^{(1)}$, where $\mathcal{W}_i^{(1)}|_{a_1}$ is constant
- Discard one-element sets $\mathcal{W}_i^{(1)}$ (as no identity involves such words)
- $a_2 \in (\mathbb{O}^\times)^n$, a new random n-tuple
- Substituting a_2 gives a finer decomposition $\mathcal{W}_1^{(2)} \cup \mathcal{W}_2^{(2)} \cup \ldots \cup \mathcal{W}_{l_2}^{(2)}$
- ... repeat until the sets $\mathcal{W}_i^{(j)}$ can no longer be refined

$v, w \in \mathcal{W}_i^{(j)} \quad \longrightarrow \quad v = w$ is a *candidate* for an identity
Attempt to prove $v = w$ in a free loop.

All identities can be found in this way.

No (nontrivial) identities in $\mathbb{O}^\times$ of degree $\leqslant d$ in n variables found for

- $d = 4$, $n = 6$
- $d = 5$, $n = 5$
- $d = 6$, $n = 3$

No identities of positive type found for

- $d = 7$, $n = 3$

An identity $v = w$ is of *positive type* (or *inverse-free*) if both v and w involve no negative exponents of the basis variables.

Positive-type identities in free Moufang loops:

- The Moufang identities (left, right, central)
- $(x.yz.x)y = xy.z.xy = x(y.zx.y)$ (two-sided)
- Consequences of the diassociativity ($xy.x = x.yx$, etc.)

Prb. Are there other positive-type identities?

• $d = 8$, $n = 3$

In this case, the following identities of positive type hold in $\mathbb{O}^\times$:

1. $b((a \cdot c^2 b \cdot a) \cdot cb) = b(a \cdot c^2 b) \cdot (a \cdot cb)$
2. $(a^2 b \cdot c)(a \cdot bcb) = (a^2 b \cdot ca) \cdot bcb$
3. $a((b \cdot ca \cdot b) \cdot c^2 a) = a(b \cdot ca) \cdot (b \cdot c^2 a)$
4. $ab \cdot (cbc \cdot a^2 b) = (ab \cdot cb)(c \cdot a^2 b)$
5. $ab \cdot (c \cdot a^2 b \cdot c)b = (ab \cdot (c \cdot a^2 b)) \cdot cb$
6. $ab \cdot ((ac \cdot b) \cdot ac^2) = a(b \cdot ac \cdot b) \cdot ac^2$
7. $a^2 b \cdot (cbc \cdot ab) = (a^2 b \cdot cb)(c \cdot ab)$

• [P.Vojtěchovský, M.Kinyon] A computer proof for idens. 1-7. (`Prover9`)

$\Rightarrow$ These identities hold in all MLs. Are they really "new"?

Moufang semiloops

"semiloop" = "groupoid" = "magma"

Moufang semiloops are sometimes defined as semiloops satisfying the Moufang identities (left,right,central).

• A free semigrp. on X is embedded into a free grp. on X, $\quad id: X \to X$.

It is natural to expect a free Moufang semiloop to be the “freest” semiloop that can be similarly embedded into a free Moufang loop.

Definition. A semiloop satisfying all identities of positive type that hold in Moufang loops is called a *Moufang semiloop*.

• Moufang semiloops are diassociative,

• satisfy the Moufang identities + identities 1–7.

An identity 1–7 is *new* if it is not a consequence (in the variety of semiloops) of the diassociativity + Moufang + the other 6 identities.

• This is true for identities 1,3–7. Unknown for identity 2. `(Mace4)`

Problem. Is the variety of Moufang semiloops finitely based?

Problem. Does every new positive-type identity contain the square of an indeterminate?

Free Moufang loops

The free Moufang loop F_n is a mysterious object. $(n \geqslant 3)$

Problem. No useful canonical form for elements of F_n. (reduced words)

Problem. Is every subloop of F_n free? (Nielsen—Schreier)

Problem. Is F_n torsion-free? (true for groups)

Problem. Is F_n Hopfian? (true for groups)

The associator-commutator series

Let M be a Moufang loop.
Commutator of $x, y \in M$: $xy = yx.c$, $c = [x, y]$
Associator of $x, y, z \in M$: $xy.z = (x.yz)a$, $a = (x, y, z)$

- $\delta_1(M) = M$,
- $\delta_n(M)$ is the normal subloop of M generated by
  - ◇ $[\delta_i(M), \delta_j(M)]$, $i, j < n$, $i + j \geqslant n$
  - ◇ $(\delta_i(M), \delta_j(M), \delta_k(M))$, $i, j, k < n$, $i + j + k \geqslant n$

$x \in \delta_i(M) \setminus \delta_{i+1}(M) \quad \to \quad x$ has *weight* i (∞, otherwise)

The normal series

$$M = \delta_1(M) \trianglerighteq \delta_2(M) \trianglerighteq \dots$$

is central: $\delta_i(M)/\delta_{i+1}(M) \leqslant Z(M/\delta_{i+1}(M))$.

The *center* $Z(M) = \{x \in M \mid [x, M] = 1, (x, M, M) = 1\}$.

The associator-commutator approximation for F_3.

• Let $F = F_3$. The abelian groups $\delta_i(F)/\delta_{i+1}(F)$ for $i = 1, 2, \dots, 5$ are free or ranks $3, 3, 9, 21, 57$, respectively.

cf. For a 3-generator free group, the ranks are $3, 3, 8, 18, 48$.

A multiplication formula for $F/\delta_6(F)$.

• $(n_1, n_2, \dots, n_5) = (3, 3, 9, 21, 57)$;
• $A = \mathbb{Z}^{n_1} \oplus \mathbb{Z}^{n_2} \oplus \dots \oplus \mathbb{Z}^{n_5}$; $\qquad x \in A$

$$x = (x_{11}, x_{12}, x_{13};\ x_{21}, x_{22}, x_{23};\ \dots, x_{ij}, \dots),$$

where $1 \leqslant i \leqslant 5$, $1 \leqslant j \leqslant n_i$.

For $x, y \in A$, we have $\quad x \star y = x + y + f$,
where $f = (f_{11}, f_{12}, f_{13}; f_{21}, \ldots, f_{ij}, \ldots)$,
f_{ij} are integer-valued polynomials in x_{rs}, y_{rs} with $1 \leqslant r < i$, $1 \leqslant s \leqslant n_r$.

f_{ij} may be expressed as multilinear polynomials in the *binomial variables* $x_{rs}^{(t)}, y_{rs}^{(t)}$, $t \geqslant 1$, where we denote

$$z^{(t)} = \frac{z(z-1)(z-2)\ldots(z-t+1)}{t!} \in \mathbb{Q}[z]$$

$f_{11}=f_{12}=f_{13}=0, \quad f_{21}=-x_{12}y_{11}, \quad f_{22}=-x_{13}y_{11}, \quad f_{23}=-x_{13}y_{12}$,

$f_{31}=-2x_{11}x_{13}y_{12}-x_{11}y_{12}y_{13}+2x_{12}x_{13}y_{11}+x_{12}y_{11}y_{13}-3x_{13}y_{11}y_{12}+6x_{23}y_{11}$,

$f_{32}=-x_{12}y_{11}^{(2)}+x_{21}y_{11}, \quad f_{33}=-x_{12}^{(2)}y_{11}-x_{12}y_{11}y_{12}+x_{21}y_{12}$,

$f_{34}=-x_{12}x_{13}y_{11}-x_{12}y_{11}y_{13}+x_{21}y_{13}-x_{23}y_{11}, \quad f_{35}=-x_{13}y_{11}^{(2)}+x_{22}y_{11}$,

$f_{36}=-x_{13}y_{11}y_{12}+x_{22}y_{12}+x_{23}y_{11}, \quad f_{37}=-x_{13}^{(2)}y_{11}-x_{13}y_{11}y_{13}+x_{22}y_{13}$,

$f_{38}=-x_{13}y_{12}^{(2)}+x_{23}y_{12}, \quad f_{39}=-x_{13}^{(2)}y_{12}-x_{13}y_{12}y_{13}+x_{23}y_{13}$

$f_{41}=-2x_{11}^{(2)}x_{13}y_{12}-x_{11}^{(2)}y_{12}y_{13}-x_{11}x_{12}y_{11}y_{13}-x_{11}x_{13}y_{11}y_{12}-x_{11}y_{11}y_{12}y_{13}$,

$-4x_{12}x_{13}y_{11}^{(2)}-3x_{12}y_{11}^{(2)}y_{13}+x_{13}y_{11}^{(2)}y_{12}+2x_{11}x_{21}y_{13}-2x_{11}x_{22}y_{12}-x_{11}y_{12}y_{22}$

$+x_{11}y_{13}y_{21}-3x_{12}x_{13}y_{11}+2x_{12}x_{22}y_{11}-2x_{12}y_{11}y_{13}+4x_{13}x_{21}y_{11}+2x_{13}y_{11}y_{12}+\ldots$

- We have $(A,\star) \cong F/\delta_6(F)$, where $F = F_3$.

Consequence: *modulo* identities

- $(a,b,c)^6 \equiv [[a,b],c]\,[[b,c],a]\,[[c,a],b] \mod \delta_4$; (cf. Hall-Witt, Jacobi)
- $(a,b,[a,c]) \equiv [(a,b,c),a] \mod \delta_5$; (cf. Mal'cev)
- $(a^n,b,c) \equiv (a,b,c)^n\,[(a,b,c),a]^{n(n-1)/2} \mod \delta_5 \quad \forall n \in \mathbb{Z}$;
- $(ab,c,d) \equiv (a,c,d)(b,c,d) \cdot [(b,c,d),a] \mod \delta_5$;
- $((a,b,c),x,y) \equiv ((a,x,y),b,c)\,((b,x,y),c,a)\,((c,x,y),a,b) \mod \delta_6$.

(the last two identities hold in 3-generator loops only)

Open problems:

Problem. Are all quotients $\delta_i(F_n)/\delta_{i+1}(F_n)$ torsion-free, $i = 1, 2, \ldots$?

Problem. Is $\bigcap_{i=1}^{\infty} \delta_i(F_n) = 1$?

Isotopy and triality

Definition. Loops M and L are *isotopic* if $\exists$ bijections $\alpha, \beta, \gamma : M \to L$ such that

$$x\alpha \cdot y\beta = (xy)\gamma \qquad \forall x, y \in M.$$

- Moufang loops M and L are isotopic iff there is $m \in M$ such that $L \cong (M, \circ_m)$, where

$$x \circ_m y = xm^{-1} \cdot my \qquad \forall x, y \in M.$$

- Let G be a group with triality $S = \langle \sigma, \rho \rangle$ and $m \in \mathcal{M}(G)$. Then G is a group with triality $S_m = \langle \sigma, \rho^2 m \rho^2 \rangle$ whose corresponding Moufang loop M_m has multiplication

$$x \star_m y = xm^{-1} \star my \qquad \forall x, y \in M_m.$$

In particular, G is a group with triality for all loop-isotopes of M.