1 Combinations, Permutations, and Elementary Probability

Roughly speaking,

Permutations are ways of grouping things where the order is important.

Combinations are ways of grouping things where the order is not important.

Suppose for example that there are three people: Ann, Bob and Carol.

Order does matter when we want to know all possible different president/vice-president pairs that can be selected from them.

But order does not matter when we want to know all possible ways of choosing two of them to go skiing during Spring break.

1.1 Factorial

Factorials are used to compute permutations and combinations. A factorial means "the product of the first n whole numbers". It is written with the exclamation sign: $n!$ and it is defined as

$$\begin{aligned} 0! &= 1 \\ 1! &= 1 \\ 2! &= 2 \times 1 = 2 \\ 3! &= 3 \times 2 \times 1 = 6 \\ 4! &= 4 \times 3 \times 2 \times 1 = 12 \\ 5! &= 5 \times 4 \times 3 \times 2 \times 1 = 60 \\ &\vdots \\ n! &= n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1. \end{aligned}$$

They grow very very fast. Notice that, by definition, $\underline{0! = 1.}$

At first glance, $\frac{16!}{15!}$ seems hard to compute:

$$\frac{16!}{15!} = \frac{16 \times 15 \times 14 \times \cdots \times 2 \times 1}{15 \times 14 \times \cdots \times 2 \times 1} = \frac{20\,922\,789\,888\,000}{1307\,674\,368\,000}.$$

However, if we reduce the fraction before we multiply it, the computation is very simple:

$$\frac{16!}{15!} = \frac{16 \times 15 \times 14 \times \cdots \times 2 \times 1}{15 \times 14 \times \cdots \times 2 \times 1} = 16.$$

Remember to simplify the fractions in the following problems:

Exercise 1 *Compute the following:*

1. $\frac{8!}{5!}$
2. $(6-3)!$
3. $6!-3!$
4. $\frac{8!}{3!5!}$
5. $\frac{100!}{97!3!}$

1.2 Permutations

These problems are solved using the multiplication principle, but sometimes we need to adjust for "over counting", like in problem 3 below.

A permutation of a set of n distinct symbols is an arrangement of them in a line in some order. There are $n!$ permutations of n symbols.

An r-permutation of n symbols is a permutation of r of them. There are $\frac{n!}{(n-r)!}$ different r-permutations of n symbols.

1. Of three people (Ann, Bob and Carol) two are selected to be president and vice-president. How many possible different president/vice-president pairs could be selected? List all possible answers and then check the solution using the multiplication principle.
2. How many different letter orderings can you make out of the word CATS? List all possible answers and then check the solution using the multiplication principle.
3. How many different orderings of the letters in the word MOON are possible? List all possible answers. Why are there fewer orderings of MOON than of CATS?
4. How many permutations of the letters in the word WONDERFUL are possible?
5. How many different arrangements of the letters in the word COLORADO are possible?
6. How many orderings of MISSISSIPPI are possible?
7. The four kids in a family are arguing over who sits where in their family car which has four passenger seats. How many possible seating arrangements are there?

1.3 Combinations

When order does not matter, the number of combinations of n things taken r at a time is

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

It is not difficult to deduce this formula. Each r combination can be arranged in $r!$ different ways. Then the number of r-permutations is equal to the number of r combinations times $r!$. Since we know that $\frac{n!}{(n-r)!}$ is the number of r-permutations, we have that

$$\binom{n}{r} r! = \frac{n!}{(n-r)!}.$$

From here we get the formula for $\binom{n}{r}$.

1. A roller coaster has 3 seats and 4 children want to ride. How many ride combinations are possible?
2. A baseball team has 13 members. How many lineups of 9 players are possible? The position of each member in the lineup is not important.
3. In how many ways can 20 students out of class of 32 be chosen to attend class on a late Thursday afternoon if
   (a) Paul refuses to go to class?
   (b) Michelle insists on going?
   (c) Jim and Michelle insist on going?
   (d) either Jim or Michelle (or both) go to class?
   (e) just one of Jim or Michele attend?
   (f) Paul and Michelle refuse to attend class together?
4. Find the number of different poker hands. A poker hand consists of 5 cards chosen from a standard pack of 52 (no jokers).
5. Find the number of two pairs of cards (e.g., $2\clubsuit, 2\diamondsuit, 6\heartsuit, 8\heartsuit, 8\spadesuit$) that you can get in a poker hand.
6. Find the number of three of a kind cards (e.g. $Q\clubsuit, Q\diamondsuit, Q\heartsuit$, $8\diamondsuit, 9\spadesuit$) that you can get in a poker hand.

1.4 Elementary Probability

If all outcomes in a finite sample space S are equally likely and A is a subset of S, then

$$P(A) = \frac{|A|}{|S|},$$

where $|A|$ is the number of elements in A and $|S|$ is the number of elements in S.

1. A pair of dice is rolled. Find the probability that the sum of the dice is 4.
2. A basket contains three red balls, four green balls, and nine yellow balls.
   (a) If one ball is drawn from the basket, what is the probability that it is yellow?
   (b) If one ball is drawn from the basket, what is the probability that it is not red?
   (c) Assume that two balls are drawn one after another. After the first ball is drawn, it is put back in the basket before the seconds ball is drawn. Find the probability that both ball are the same color.
   (d) Repeat $1.c$ assuming the first ball is not put back in the basket before the seconds one is drawn.
3. Three dice are rolled.
   (a) Find the probability that each of the three numbers is a multiple of 5.
   (b) Find the probability that the three numbers are different.

1.5 Mixed Problems

1. Twelve first year seminar students line up for a fire drill.
   (a) How many possible arrangements are there?
   (b) How many arrangements have David and Steph next to each other?
   (c) If they line up at random, what is the probability that David and Steph will be next to each other?
2. Eight people met at a New Year's Eve party and all shake hands. How many handshakes were there?
3. Teacher Thelma says *"You may work these five problems in any order you choose."* There are 30 students in the class. Is it possible for all 30 students to work the problems in a different order? Justify your answer. Don't just answer yes or no.
4. In a popular lottery known as 6/49, a player marks a card with six different numbers from 1 to 49 and wins if the numbers match six randomly selected such numbers.
   (a) In how many ways can a player complete a game card?
   (b) How many cards need to be completed to have at least one chance in a million of winning?
   (c) In how many ways can a player complete a game card so that

    i. no number matches any of the 6 selected.
    ii. exactly one number matches one of the 6 selected.
    iii. exactly two numbers match two of the 6 selected.
    iv. all 6 numbers match.

  (d) Find the probability of having
    i. exactly three of the numbers matching the selected numbers.
    ii. at most five of the numbers matching the selected numbers.

5. Find the probability of that a three digit number selected at random from 000 to 999 (inclusive) has all digits different.

6. Wanda is going to toss a coin eight times. In how many ways can she get five heads and three tails?

7. In how many ways can 10 adults and 5 children stand in a line so that no two children are next to each other?

8. How many 12-digit $0-1$ strings contain precisely five 1's.

9. How many ways are there to distribute eight different books among thirteen people if no person is to receive more than one book?

10. Answer the following problems
    (a) In how many ways can 10 boys and 4 girls sit in a row?
    (b) In how many ways can they sit in a row if the boys sit together and the girls sit together?
    (c) In how many ways can they sit in a row if the girls are to sit together?
    (d) In how many ways can they sit in a row if only the girls are to sit together?

11. Eight horses enter a race in which first, second, and third place will be awarded. Assuming no ties, how many different outcomes are possible?

12. How many dance partners of different gender can be selected from a group of 12 women and 20 men?

13. What is the probability that a card drawn at random from a shuffled deck of 52 normal playing cards is a Heart or a Face card? Be sure to avoid double counting!

14. A 6-letter permutation is selected at random from the letters UNITED. What is the probability that:
    (a) The third letter is I and the last letter is T?
    (b) The second letter is a vowel and the third is a consonant?

(c) The second and third letters are both vowels?

(d) The second letter is a consonant and the last letter is E?

(e) The second letter is a consonant and the last letter is T?

15. Find the number of poker hands of each type. For the purposes of this problem, a poker hand consists of 5 cards chosen from a standard pack of 52 (no jokers). Also for the purposes of this problem, the ace can only be a high card. In other words, the card sequence $A\clubsuit, 2\diamondsuit, 3\diamondsuit, 4\spadesuit, 5\heartsuit$ is not a straight, since the ace is a high card only. Here are the definitions of the hands:

(a) Royal flush: 10 through A in the same suit; e.g, $10\clubsuit, J\clubsuit, Q\clubsuit, K\clubsuit, A\clubsuit$.

(b) Straight flush: 5 cards in sequence in the same suit, but not a Royal flush; e.g., $8\clubsuit, 9\clubsuit, 10\clubsuit, J\clubsuit, Q\clubsuit$.

(c) Four of a kind: Four cards of the same rank; e.g., $Q\clubsuit, Q\diamondsuit, Q\heartsuit, Q\spadesuit, 8\spadesuit$.

(d) Full house: Three cards of one rank and two of another ; e.g., $Q\clubsuit, Q\diamondsuit, Q\heartsuit,\ 8\diamondsuit, 8\spadesuit$.

(e) Flush: Five cards in the same suit that are not in sequence; e.g., $2\clubsuit, 4\clubsuit, 6\clubsuit, 8\clubsuit, 10\clubsuit$.

(f) Straight: Five cards in sequence that are not all in the same suit; e.g., $8\clubsuit, 9\diamondsuit, 10\diamondsuit, J\clubsuit, Q\clubsuit$.

(g) Three of a kind: Three cards of the same rank; the others of different rank; e.g., $Q\clubsuit, Q\diamondsuit, Q\heartsuit,\ 8\diamondsuit, 9\spadesuit$.

(h) Two pairs: Two pairs of cards; e.g., $2\clubsuit, 2\diamondsuit, 6\heartsuit, 8\heartsuit, 8\spadesuit$.

(i) Pair: A single pair of cards; e.g., $2\clubsuit, 2\diamondsuit, 6\heartsuit, 8\heartsuit, 10\spadesuit$.

(j) Bust: A hand with none of the above; e.g, $2\clubsuit, 4\diamondsuit, 6\heartsuit, 8\heartsuit, 10\spadesuit$.