Example (§ 3.9)

Suppose $X_1, \ldots X_n$ are iid and $X_i \sim N(\mu, \sigma^2)$ and σ^2 is known. Given outcomes $(x_1, x_2, \ldots x_n)$ for $(X_1, X_2, \ldots X_n)$, calculate a symmetric confidence interval that contains μ with probability $1-\alpha$.

$$\frac{1}{n}\sum_{i=1}^{n} X_i \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

Find $(-b, b)$ such that

$$P(-b \leq Z \leq b) = 1-\alpha$$ where Z is the standard normal random variable.

(Use the table to find b with $P(Z \leq b) = 1-\alpha/2$)

Now

$$P\left(-b \leq \frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mu}{\frac{\sigma}{\sqrt{n}}} \leq b\right) = 1-\alpha$$

so $$P\left(\frac{-b\sigma}{\sqrt{n}} \leq \bar{X} - \mu \leq \frac{b\sigma}{\sqrt{n}}\right) = 1-\alpha$$

$$P\left(\frac{-b\sigma}{\sqrt{n}} \leq \mu - \bar{X} \leq \frac{b\sigma}{\sqrt{n}}\right) = 1-\alpha$$

$$P\left(\bar{X} - \frac{b\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + \frac{b\sigma}{\sqrt{n}}\right) = 1-\alpha$$

Report a confidence interval

$$\left(\bar{x} - \frac{b\sigma}{\sqrt{n}}, \bar{x} + \frac{b\sigma}{\sqrt{n}}\right)$$ for μ

For a numerical example, suppose we know that a process to cut lengths of wire in lengths with a given nanosecond traversal time produces lengths that independent and normally distributed with variance $0.004\ ns^2$. A sample of 100 wires has mean 'length' 50.02 ns. Give a 99% confidence interval for the mean length of wires produced by this process.

$P(Z \le 2.58) = .9951$, so we'll use $b = 2.58$

The interval in question is $\approx$

$$\left(50.02 - \frac{\sigma}{\sqrt{100}} b,\ 50.02 + \frac{\sigma}{\sqrt{100}} b\right) =$$

$$\left(50.02 - \frac{.02}{100}(2.58),\ 50.02 + \frac{.02}{100}(2.58)\right)$$

$$(50.015,\ 50.025)$$