Higher Signature on Witt Spaces

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Poincaré duality and classical signature

Let M be an oriented closed manifold of dimension n.

Poincaré Duality

$H^k(M) \times H^{n-k}(M) \to \mathbb{C}$ is nondegenerate bilinear for $0 \leq k \leq n$.

If $\dim M = 4k$, this gives a symmetric bilinear form on

$$H^{2k}(M) \times H^{2k}(M) \to \mathbb{C}.$$

Definition

$\mathrm{sign}(M) :=$ the signature of this symmetric bilinear form.

Spaces with singularities

If X is a space with singularities, then Poincaré duality fails in general. For example, think of $X = S^2 \vee S^2$, where $H^0(X) = \mathbb{C}$ and $H^2(X) = \mathbb{C} \oplus \mathbb{C}$.

The failure of Poincaré duality here is due to the presence of singularities. Goresky and MacPherson (1978) introduced intersection homology theory, and proved a generalized Poincaré duality for a class of spaces with singularities, called pseudomanifolds.

Pseudomanifolds

Definition (Pseudomanifold)

A (p.l.) pseudomanifold of dimension n is a locally compact space X containing a closed subspace Σ with $\dim(\Sigma) \leq n-2$ such that $X - \Sigma$ is an n-dimensional oriented manifold which is dense in X.

A pseudomanifold X admits a stratification

$$X = X_n \supset X_{n-1} = X_{n-2} \supset X_{n-3} \supset \cdots \supset X_0$$

where $X_j - X_{j-1}$ is a manifold of dimension j, if nonempty. Think of a triangulation of X, and

$$X_n = |T_n| \supset X_{n-2} = |T_{n-2}| \supset \cdots \supset X_0 = |T_0|$$

Perversity and intersection homology

Definition (Perversity)

Given a pseudomanifold X of dimension n, a perversity, denoted by $\bar{p}$, is a sequence of integers

$$\bar{p} = (p_2, p_3, \cdots, p_n)$$

with $p_2 = 0$ and $p_{k+1} = p_k$ or $p_k + 1$.

Minimum perversity $\bar{0} = (0, 0, \cdots, 0)$, and maximum perversity $\bar{t} = (0, 1, 2, \cdots, n-2)$.

Perversity is used to prescribe "transversality condition" of how simplices intersecting the singular strata of X.

Chain complexes for intersection homology

Definition

Fix a perversity $\bar{p}$. A subspace $Y \subset X$ is called $(\bar{p}, i)$-allowable, if $\dim Y \leq i$ and $\dim(Y \cap X_{n-k}) \leq i - k + p_k$.

Now we define a simplicial chain complex for intersection homology.

Definition

$IC_i^{\bar{p}}(X)$ = all simplices ξ such that ξ is $(\bar{p}, i)$-allowable and $\partial\xi$ is $(\bar{p}, i-1)$-allowable.

Generalized Poincaré duality

Theorem (Goresky & MacPherson 1978)

Given an oriented pseudomanifold X of dimension n, then

$$IH_i^{\bar{p}}(X) \times IH_{n-i}^{\bar{q}}(X) \to \mathbb{C}$$

is nondegenerate, where $\bar{p} + \bar{q} = \bar{t}$.

In particular, if $\dim X = 4k$, then

$$IH_{2k}^{\bar{m}}(X) \times IH_{2k}^{\bar{n}}(X) \to \mathbb{C}$$

where $\bar{m} = (0, 0, 1, 1, 2, 2, \cdots)$ and $\bar{n} = (0, 1, 1, 2, 2, 3, \cdots)$ are lower middle and upper middle perversities.

Generalized signature

There is a natural map $IH_j^{\bar{m}}(X) \to IH_j^{\bar{n}}(X)$. However, this map is *not* an isomorphism in general.

Generalized signature for Witt spaces

If X is a Witt space, then $IH_j^{\bar{m}}(X) \cong IH_j^{\bar{n}}(X)$.

$$\mathrm{sign}(X) = \text{signature of the quadratic form on } IH_{2k}^{\bar{p}}(X)$$

Definition (Siegel 1983)

A pseudomanifold X is a Witt space, if for each p in an *odd-codimensional* stratum, the middle-dim intersection homology group of the link of p vanishes.

Higer Signature on smooth manifolds

FACT

classical signature of M = the index of the signature operator on M

$$0 \to \Omega^0_{L^2}(M) \xrightarrow{d} \Omega^1_{L^2}(M) \xrightarrow{d} \cdots$$

where d is the de Rham differential. Combined with the Hodge star operator, we get the signature operator D.

$D \rightsquigarrow K$-homology class in $K_n(M)$, whose higher index class in $K_n(C_r^*(\Gamma))$ is called the higher signature of M. Here $n = \dim M$ and $\Gamma = \pi_1(M)$.

Higher Signature on Witt spaces (analytic approach)

For a Witt space X, Cheeger (83) defined the signature operator by imposing certain metrics of conic type (on the regular part of X). The K-homology class is independent of the metric.

Its Chern character (or $\mathcal{L}$-class) were studied by Siegel (83) and Moscovici-Wu (97).

Cheeger's approach was further developed by Albin, Leichtnam, Mazzeo and Piazza (2012). They allow more general metrics on the regular part of X.

Higher Signature on Witt spaces (analytic approach)

Theorem (Albin, Leichtnam, Mazzeo and Piazza (2012))

X a Witt space of dimension n with $\pi_1(X) = \Gamma$, then the K-homology $[D] \in K_n(X)$ is independent of the metric.

(1) $\mathrm{ind}_\Gamma(D) \in K_n(C_r^*(\Gamma))$ *is a cobordism invariant.*

(2) $\mathrm{ind}_\Gamma(D) \in K_n(C_r^*(\Gamma))$ *is a stratified-homotopy invariant.*

Noncommutative geometric approach

We consider a more conceptual approach follows the work of Mishchenko, Ranicki, and Higson&Roe.

An n-dimensional Hilbert-Poincaré complex (over a C^*-algebra A) is a complex of finitely generated Hilbert A-modules

$$E_0 \xleftarrow{b_1} E_1 \xleftarrow{b_2} \cdots \xleftarrow{b_n} E_n$$

together with adjointable operators $T : E_p \to E_{n-p}$ such that

(1) if $v \in E_p$, then $T^* v = (-1)^{(n-p)p} T v$;

(2) if $v \in E_p$, then $Tb^*(v) + (-1)^p bT(v) = 0$;

(3) T induces an isomorphism from the homology of the dual complex

$$E_n \xleftarrow{b_n^*} E_{n-1} \xleftarrow{b_{n-1}^*} \cdots \xleftarrow{b_1^*} E_0$$

to the homology of the complex (E, b).

Noncommutative geometric approach

Given a Witt space X, we show that the chain complex $IC_i^{\bar{m}}(X)$ (after completion) gives rise to such a Hilbert-Poincaré complex.

(i) for K-homology, we work with $IC_i^{\bar{m}}(\mathcal{C}(X))$, where $\mathcal{C}(X)$ is the coarse geometric cone of X.

(ii) for higher signature, we work with $IC_i^{\bar{m}}(\widetilde{X})$ completed to a Hilbert-module over $C_r^*(\Gamma)$, where $\widetilde{X}$ is the universal cover of X and $\Gamma = \pi_1(M)$.

Noncommutative geometric approach

Theorem (Higson, X. 14)

X a Witt space of dimension n. Then the above chain complexes define the signature K-homology class and higher signature of X. Moreover, in this framework, various invariance properties of the higher signature, such as cobordism invariance and stratified-homotopy invariance, are automatic.

Thank you!