Research Statement

My main research areas are probability and mathematical physics (with emphasis on statistical physics). A continuing theme of my research is applying renormalization group ideas to study relevant problems arising in physics and probability, and in particular to develop a quantitative theory of phase transitions. Besides extending my dissertation research on the spectral properties of the renormalization group for Ising systems, my postdoctoral research has branched out to examine the phase structure of exponential random graphs. I am also interested in establishing the connection between random graphs and Ising systems and in introducing renormalization group ideas to random graph models.

My work on renormalization focuses on the existence and properties of the renormalization group linearization for Ising-type lattice spin systems. A renormalization group transformation consists of averaging over short distance lattice blocks, followed by a rescaling that maps each block into a single lattice site. There is the astonishing empirical fact that certain exponents associated with critical phenomena are universal, and are related to eigenvalues of the linearized renormalization group map near the nontrivial fixed point. The renormalization group transformation may be thought of as a map of potentials that define the Gibbs state. The linearization then should be a linear transformation acting on such a Banach space. The first part of my work carries out some explicit calculations of the spectrum of the linearization of the renormalization group at infinite temperature and discovers that it is of an unusual kind. The second part treats the rigorous definition of the renormalization group map in the infinite volume limit at high temperature by a cluster expansion approach. The third part justifies the differentiability of the renormalization group map in the infinite volume limit at the critical temperature under a certain condition.

My work on random graphs centers on the structure and behavior of the exponential family of random graphs, which is among the most promising class of network models. Dependence between the random edges is defined through certain finite subgraphs, in imitation of the use of potential energy to provide dependence between particle states in a grand canonical ensemble of statistical physics. By adjusting the specific values of these subgraph densities, one could analyze the influence of various local features on the global structure of the network. Loosely put, a phase transition occurs when a singularity arises in the limiting free energy, as it is the generating function for the limiting expectations of all thermodynamic observables. The first part of my work derives the full phase diagram for a large family of 2-parameter exponential random graph models with attraction, each containing a first order transition curve ending in a second order critical point. The second part extends the phase picture to the 3-parameter case and shows that it consists of a first order surface phase transition bordered by a second order critical curve. The third part examines the asymptotic structure of 2-parameter exponential random graph models with repulsion and observes quantized behavior along general straight lines. The fourth part considers generic k-parameter families of exponential random graphs from a lattice gas (Ising) perspective, making the exponential random graph model treatable by cluster expansion techniques.

The following sections describe my research in more detail.

Spectral properties of the renormalization group at infinite temperature [22] Israel [13] found the operator bound of the linearization of the renormalization group map at infinite temperature (zero interaction) corresponding to decimation and majority rule in a Banach algebra setting, but did not go into detail about the spectral type of this transformation. We delve more into this matter and analyze the spectrum of these renormalization group operations completely. It is of an unusual kind: dense point spectrum for which the adjoint operators have no point spectrum at all, but only residual spectrum. This may serve as a lesson in what one might expect in more general situations.

A cluster expansion approach to renormalization group transformations [23] We treat the rigorous definition of the renormalization group map at high temperature. A cluster expansion [4, 11, 15] is used to justify the existence of the partial derivatives of the renormalized interaction with respect to the original interaction. This expansion is derived from the formal expressions, but it is itself well defined and convergent. Suppose in addition that the original interaction is finite range and translation invariant. We show that the matrix of partial derivatives in this case displays an approximate band property. This in turn gives an upper bound for the renormalization group linearization.

Renormalization group transformations near the critical point: Some rigorous results [24] The real interest of the renormalization group is to define the transformation at the critical temperature [9]. With the help of the Dobrushin uniqueness condition [8] and standard results on the polymer expansion, Haller and Kennedy [12] provided a sufficient condition for the existence of the renormalized Hamiltonian in a neighborhood of the critical point. By a more complicated but reasonably straightforward application of the cluster expansion machinery, we show that their condition would further imply a band structure on the matrix of partial derivatives of the renormalized interaction with respect to the original interaction.

Phase transitions in exponential random graphs [18] In their recent paper [6], Chatterjee and Diaconis gave the first rigorous proof of singular behavior in a specific exponential random graph model, the edge-triangle model, using the emerging tools of graph limits as developed by Lovász and coworkers [16]. We extend their result both in the class of models and parameter values under control and concentrate on the phenomenon of phase transitions. By carrying out a detailed analysis of a maximization problem, we derive the full phase diagram for a large family of 2-parameter exponential random graph models with attraction, each containing a first order transition curve ending in a second order critical point, qualitatively similar to the gas/liquid transition in equilibrium materials.

Critical phenomena in exponential random graphs [25] One of the key motivations for studying exponential random graphs is to develop models that exhibit transitivity and clumping (i.e., a friend of a friend is likely also a friend). However, as seen in experiments and through heuristics [17], it is often not enough to have only 2 subgraphs as sufficient statistics. To accurately model the global structure of the network, more local features of the random graph need to be captured [6]. We therefore incorporate the density of one more subgraph into the probability distribution and extend the phase picture of the attractive exponential model to the 3-parameter case. We find that it consists of a first order surface phase transition bordered by a second order critical curve. This phase picture may be further extended to a k-parameter setting, and the parameter space is expected to comprise a single phase with first order phase transitions across one (or more) hypersurfaces and second order phase transitions along their boundaries.

Asymptotic quantization of exponential random graphs [26] The phase transition in the repulsive exponential random graph model may also be characterized by solving a maximization problem. However, due to its complicated structure, this maximization problem is not always explicitly solvable. For the edge-triangle model, Chatterjee and Diaconis [6] showed that the graph becomes roughly bipartite when the edge parameter is fixed and the triangle parameter is large and negative, i.e., when the two parameters trace a vertical line downward. In hope of observing more interesting extremal behaviors, we investigate the asymptotic structure of this classic model along general straight lines. We show that as we continuously vary the slopes of these lines, the graph exhibits quantized behavior, jumping from one complete multipartite graph to another, and the jumps happen precisely at the normal lines of an infinite polytope. More intriguingly, this quantized behavior in the repulsive region could be further tied to the jump discontinuity in graph density in the attractive region [6, 18]. Similar results are expected for more general repulsive models.

A cluster expansion approach to exponential random graph models [27] We show that any k-parameter exponential random graph model may alternatively be viewed as a lattice gas model with a finite Banach space norm. The system may then be treated by cluster expansion methods [15] from statistical physics. In particular, we derive a convergent power series expansion for the limiting free energy in the case of small parameters. Since the free energy is the generating function for the expectation values of other random variables, this characterizes the structure and behavior of the limiting network in this parameter region.

Ongoing & future research

(1) Concerning renormalization: We have shown the existence of the renormalization group linearization for Ising systems under various circumstances [23, 24], but we have not been able to show that it lies in a Banach space (except for one with uniform norm). I hope to overcome this difficulty by performing more involved cluster expansions. My more ambitious goal is to prove actual Gâteaux differentiability (or even Fréchet differentiability) of the renormalization group transformation, which would require examining higher order expansion terms. Another possible generalization is in the context of Potts models on hierarchical lattices [20]. Working with transmissivities rather than coupling constants, similar existence results as in the Ising case are expected. My future plans also include verifying the existence of the renormalization group map under more relaxed conditions, so that it can be defined iteratively, as widely assumed in physics [21].

(2) Concerning random graphs: The main hope for progress in network analysis seems to lie in formulating a model based on some alternative ensemble of graph structures. I am studying a group of papers on graphs uniformly distributed with a given degree sequence [2, 7] and hope they would provide some inspiration. Chatterjee and Diaconis [6] suggested that models with repulsion quite generally exhibit a transition qualitatively like the solid/fluid transition, in which one phase has nontrivial structure, as distinguished from the disordered Erdős-Rényi graphs [10], which have independent edges. The existence of such a transition in 2-parameter models has been proved by Aristoff and Radin [1] and Yin [28]. Some progress has recently been made in the asymptotic region [26]. However, the network structure below the transition line is still not well understood. I hope to resolve this issue and make some headway in the sparse case using methods from [3]. I would also like to explore other notions of convergence besides the graphon convergence, examine the connections between random graphs and Ising models more in-depth, and use renormalization group analysis to study random graph models.

(3) New lines of research: My research focus, statistical physics, aims at describing large scale properties of physical systems based on models which specify interactions at the atomic level. We typically model physical systems by random combinatorial structures, such as graphs and lattices. Tools used come from a wide range of areas: analysis, combinatorics, discrete geometry, probability, mathematical physics, and discrete math. Because of their strong resonance with the intertwined concepts of conformality and universality, isoradial graphs (as coined by Kenyon [14]) give, for many models of statistical physics, the right framework for studying models at criticality. I am interested to expand my research towards this area. The worlds of discrete and continuous math are not very far apart, and I have been thinking about applying renormalization group ideas to differential equations. One direction [19] is to derive exact differential renormalization flow equations and then analyze its solution about the local fixed point. Another direction [5] is to study large time behavior of solutions to nonlinear parabolic equations by perturbative epsilon expansion.

(4) Vertical integration: I am a believer in the philosophy of vertical integration, in which the research enterprise is shared among undergrads, grad students, postdocs, and faculty. In this spirit I am attracted to the idea of designing an independent study project for an appropriate student. For instance, it could be a year-long project involving random graph theory and applications. The first semester I would familiarize the student with background material. The second semester I would help the student choose a project and get a start on research. This would culminate in a report or paper that explains the material and presents the results of the research. There are many possible approaches, depending on the interests and level of preparation of the student. These range from exploration via computer visualization to rigorous probability to modeling in the context of a particular application. Thus the subject has appeal that works on all levels of the vertical integration framework.

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