Phase transitions in the edge-triangle exponential random graph model

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April 28, 2014

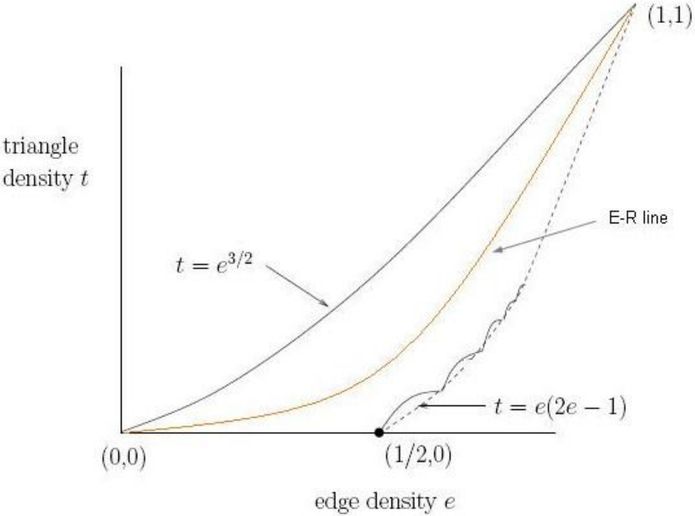
Probability space: The set $\mathcal{G}_n$ of all simple graphs G_n on n vertices.
Probability mass function:

$$\mathbb{P}_n^{\beta}(G_n) = \exp\left(n^2(\beta_1 e(G_n) + \beta_2 t(G_n) - \psi_n^{\beta})\right).$$

$e(G_n) = \frac{2|E(G_n)|}{n^2}$ and $t(G_n) = \frac{6|T(G_n)|}{n^3}$ are the edge and triangle densities of G_n.
Normalization constant ψ_n^{β}:

$$\psi_n^{\beta} = \frac{1}{n^2} \log \sum_{G_n \in \mathcal{G}_n} \exp\left(n^2(\beta_1 e(G_n) + \beta_2 t(G_n))\right).$$

Survey: Fienberg, Introduction to papers on the modeling and analysis of network data I & II. arXiv: 1010.3882 & 1011.1717.



(Razborov and others)

$\beta_2 = 0$:

$$\begin{aligned}\mathbb{P}_n^{\beta}(G_n) &= \exp\left(n^2(\beta_1 e(G_n) - \psi_n^{\beta})\right) \\ &= \exp\left(2\beta_1 E(G_n) - n^2\psi_n^{\beta}\right).\end{aligned}$$

Erdős-Rényi graph $G(n, \rho)$,

$$\mathbb{P}_n^{\rho}(G_n) = \rho^{E(G_n)}(1-\rho)^{\binom{n}{2} - E(G_n)}.$$

Include edges independently with parameter $\rho = e^{2\beta_1}/(1 + e^{2\beta_1})$.

$$\exp(n^2\psi_n^{\beta}) = \sum_{G_n \in \mathcal{G}_n} \exp\left(2\beta_1 E(G_n)\right) = \left(\frac{1}{1-\rho}\right)^{\binom{n}{2}}.$$

What happens when $\beta_2 \neq 0$?
Problem: Graphs with different numbers of vertices belong to different probability spaces!
Solution: Theory of graph limits (graphons)! (Lovász and coauthors; earlier work of Aldous and Hoover)
Graphon space $\mathcal{W}$ is the space of all symmetric measurable functions $h(x, y)$ from $[0, 1]^2$ into $[0, 1]$. The interval $[0, 1]$ represents a 'continuum' of vertices, and $h(x, y)$ denotes the probability of putting an edge between x and y.

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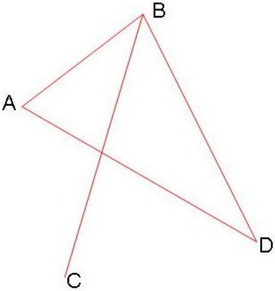
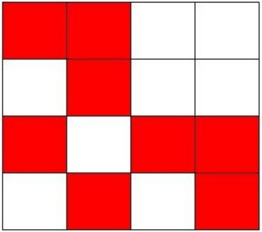
Example: Erdős-Rényi graph $G(n, \rho)$, $h(x, y) = \rho$.
Example: Any $G_n \in \mathcal{G}_n$,

$$h(x, y) = \begin{cases} 1, & \text{if } (\lceil nx, ny \rceil) \text{ is an edge in } G_n; \\ 0, & \text{otherwise.} \end{cases}$$





Large deviation:

$$\lim_{n\to\infty} \psi_n^\beta = \max_{h\in\mathcal{W}} \left(\beta_1 e(h) + \beta_2 t(h) - \iint_{[0,1]^2} I(h)dxdy \right),$$

where:

$$e(h) = \int_{[0,1]} h(x,y)dxdy,$$

$$t(h) = \iiint_{[0,1]^3} h(x,y)h(y,z)h(z,x)dxdydz,$$

and $I : [0,1] \to \mathbb{R}$ is the function

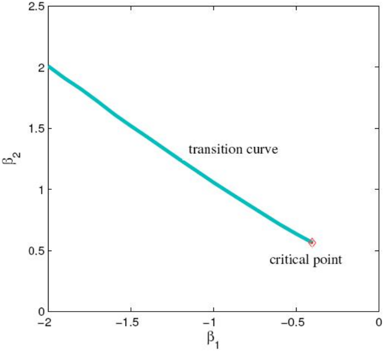
$$I(u) = \frac{1}{2}u\log u + \frac{1}{2}(1-u)\log(1-u).$$

Let F^* be the set of maximizers. G_n lies close to F^* with high probability for large n. (Chatterjee and Varadhan; Chatterjee and Diaconis)

$\beta_2 \geq 0$: Let $n \to \infty$. G_n behaves like the Erdős-Rényi graph $G(n, u^*)$, where $u^* \in [0,1]$ maximizes

$$l(u) = \beta_1 u + \beta_2 u^3 - \frac{1}{2} u \log u - \frac{1}{2}(1-u)\log(1-u).$$

(Chatterjee and Diaconis; Häggström and Jonasson; Bhamidi, Bresler, and Sly)

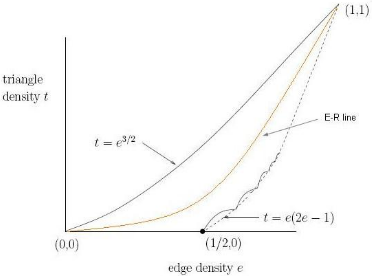


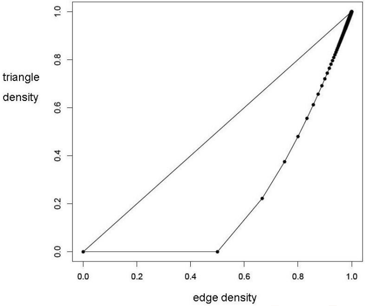
Critical point is $(\frac{1}{2}\log 2 - \frac{3}{4}, \frac{9}{16})$. (Radin and Y)

Fix β_1. Let $n \to \infty$ and then let $\beta_2 \to -\infty$. G_n looks like a complete bipartite graph with $\frac{1}{1+e^{2\beta_1}}$ fraction of edges randomly deleted. (Chatterjee and Diaconis)

Take $\beta_1 = a\beta_2 + b$. Fix a and b. Let $n \to \infty$ and then let $\beta_2 \to -\infty$. G_n exhibits quantized behavior, jumping from one complete multipartite graph to another, and the jumps happen precisely at the normal lines of an infinite polytope.

Idea: Minimize $ae + t$. (Y, Rinaldo, and Fadnavis; related work in Handcock; Rinaldo, Fienberg, and Zhou)

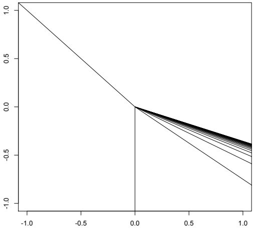


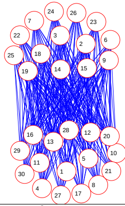


What happens if we interchange the limits in n and β_2?

Discontinuous transition along critical directions.

Idea: Study the induced probability distribution on $S_n = \{(e(G_n), t(G_n)), G_n \in \mathcal{G}_n\} \subset [0,1]^2$.





$\beta_1 = 60$, $\beta_2 = -110$, $n = 30$

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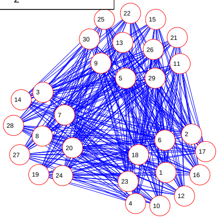

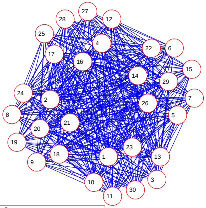

$\beta_1 = 80$, $\beta_2 = -40$, $n = 30$

Fix e. What happens if we only consider graphs whose edge density is close to e, say $|e(G_n) - e| < \alpha$?
(conditional) Probability mass function:

$$\mathbb{P}_{n,\alpha}^{e,\beta}(G_n) = \exp\left(n^2(\beta t(G_n) - \psi_{n,\alpha}^{e,\beta})\right).$$

$t(G_n) = \frac{6|T(G_n)|}{n^3}$ is the triangle density of G_n.
(conditional) Normalization constant $\psi_{n,\alpha}^{e,\beta}$:

$$\psi_{n,\alpha}^{e,\beta} = \frac{1}{n^2} \log \sum_{G_n \in \mathcal{G}_n : |e(G_n) - e| < \alpha} \exp\left(n^2 \beta t(G_n)\right).$$

Large deviation:

$$\lim_{\alpha \to 0} \lim_{n \to \infty} \psi_{n,\alpha}^{e,\beta} = \max_{h \in \mathcal{W}: e(h) = e} \left(\beta t(h) - \iint_{[0,1]^2} I(h) dxdy \right),$$

where:

$$e(h) = \int_{[0,1]} h(x,y) dxdy,$$

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Let F^* be the set of maximizers. G_n lies close to F^* with high (conditional) probability for large n. (Kenyon and Y)

Fix e and t. Consider graphons with $e(h) = e$ and $t(h) = t$ that minimize $\iint_{[0,1]^2} I(h)dxdy$. Why are we interested?
They are useful for studying exponential models without constraints:

$$\max_{h \in \mathcal{W}} \left(\beta_1 e(h) + \beta_2 t(h) - \iint_{[0,1]^2} I(h)dxdy \right)$$

$$= \max_{e,t} \max_{h \in \mathcal{W}: e(h)=e, t(h)=t} \left(\beta_1 e + \beta_2 t - \iint_{[0,1]^2} I(h)dxdy \right),$$

and with constraints:

$$\max_{h \in \mathcal{W}: e(h)=e} \left(\beta t(h) - \iint_{[0,1]^2} I(h)dxdy \right)$$

$$= \max_{t} \max_{h \in \mathcal{W}: e(h)=e, t(h)=t} \left(\beta t - \iint_{[0,1]^2} I(h)dxdy \right).$$

Special strip: Fix $e = \frac{1}{2}$ and $t \leq e^3$. Graphon $h(x, y)$ that minimizes $\iint_{[0,1]^2} I(h)dxdy$ is given by

$$h(x, y) = \begin{cases} \frac{1}{2} + \epsilon, & \text{if } x < \frac{1}{2} < y \text{ or } x > \frac{1}{2} > y; \\ \frac{1}{2} - \epsilon, & \text{if } x, y < \frac{1}{2} \text{ or } x, y > \frac{1}{2}, \end{cases}$$

where $0 \leq \epsilon = (e^3 - t)^{\frac{1}{3}} \leq \frac{1}{2}$. (Radin and Sadun)
Then

$$\max_{h \in \mathcal{W}: e(h) = e} \left(\beta t(h) - \iint_{[0,1]^2} I(h)dxdy \right)$$

$$= \max_{\epsilon \in [0, \frac{1}{2}]} \left(\beta(e^3 - \epsilon^3) - I(\frac{1}{2} + \epsilon) \right).$$

As β decreases from 0 to $-\infty$, G_n jumps from being Erdős-Rényi to almost complete bipartite (with $\epsilon > 0.47$) at $\beta \approx -2.7$, skipping a large portion of the $e = \frac{1}{2}$ line. (Kenyon and Y)

Fix β_1 and β_2. (macro) Euler-Lagrange equation for the maximizer:

$$\beta_1 + 3\beta_2 \int_{[0,1]} h(x,z)h(y,z)dz = \frac{1}{2}\log\frac{h(x,y)}{1-h(x,y)}.$$

(Chatterjee and Diaconis)
Fix e and t. (micro) Euler-Lagrange equation for the maximizer:

$$\int_{[0,1]} h(x,z)h(y,z)dz = \lambda + \mu\log\frac{h(x,y)}{1-h(x,y)}.$$

Plus: Stronger! Holds for the entire (e,t)-space.
Minus: The relationship between e, t and λ, μ is not as explicit.
(Kenyon and Y)

Thank You!

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