MacNeille Completions of FL-algebras

Agata Ciabattoni, Nikolaos Galatos, and Kazushige Terui

Abstract. We show that a large number of equations are preserved by Dedekind-MacNeille completions when applied to subdirectly irreducible FL-algebras/residuated lattices. These equations are identified in a systematic way, based on proof-theoretic ideas and techniques in substructural logics. It follows that a large class of varieties of Heyting algebras and FL-algebras admits completions.

1. Introduction

Completions of ordered algebras have been studied quite extensively, see e.g., [10] for a survey. Two typical examples are Dedekind's completion of the ordered ring of rational numbers to that of extended real numbers (i.e., real numbers with $\pm\infty$) and canonical extensions of Boolean algebras. Among the various completions, of particular importance are the generalizations of Dedekind's completion, known in the literature as *MacNeille completions* or *Dedekind-MacNeille completions*, since they are *regular*, i.e., they preserve all existing joins and meets.

In [3], Bezhanishvili and Harding proved the striking fact that there are only three subvarieties of the variety HA of Heyting algebras closed under MacNeille completions: the trivial variety, the variety BA of Boolean algebras, and the whole variety HA. This means that no equation defining an intermediate variety between BA and HA is preserved by MacNeille completions; prelinearity $(x \to y) \vee (y \to x) = 1$ is such an example. However, it is known that prelinearity *is* preserved by MacNeille completions, when applied to *subdirectly irreducible* Heyting algebras; this is because prelinear such algebras are chains, and the MacNeille completion of a chain is clearly a chain. As Heyting chains generate the variety GA of Gödel algebras, we may deduce that GA, an intermediate variety between BA and HA, admits completions, though not exactly MacNeille.

The purpose of this note is to show that this phenomenon applies to a wide class of such equations, lying in a specific level of a syntactic hierarchy of equations. We present our results in the more general setting of the variety FL

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of *FL-algebras* (also known as *pointed residuated lattices* [7]), which contains HA, as well as other interesting algebras including MV-algebras and lattice-ordered groups.

Our work is part of *algebraic proof theory*, a research project that aims to explore the connections between order algebra and proof theory. In that respect the motivation and the intuition behind these completions stems from proof-theoretic considerations. In [4, 5] we introduced the *substructural hierarchy* $(\mathcal{N}_n, \mathcal{P}_n)_{n\in\mathbb{N}}$ which naturally classifies FL-equations, in analogy to the arithmetical hierarchy; see Figure 1. In [5] we proved that the equations in the lowest classes of the hierarchy (i.e., up to $\mathcal{N}_2$) are preserved by MacNeille completions when applied to FL_w-algebras (integral and bounded FL-algebras). Moreover when these equations satisfy an additional syntactic condition (acyclicity), they are preserved by MacNeille completions when applied to arbitrary FL-algebras.

In this paper we investigate MacNeille completions for equations in $\mathcal{P}_3$–the next level of the substructural hierarchy. Equations in $\mathcal{P}_3$ include prelinearity, weak excluded middle, n-excluded middle, weak nilpotent minimum, (sm) (defining the least non-trivial, non-Booloean subvariety of HA), and many others. We show that all $\mathcal{P}_3$ equations are preserved by MacNeille completions when applied to *subdirectly irreducible* FL_{ew}-algebras (i.e., commutative FL_w-algebras). This implies that subvarieties of $\mathsf{FL_{ew}}$ (and hence subvarieties of HA) defined by these equations admit completions. Furthermore we prove a similar result for FL-algebras in general, by using the syntactic condition of acyclicity and suitably modifying the definition of the class $\mathcal{P}_3$.

2. Preliminaries

2.1. FL-algebras. A *residuated lattice* is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$, such that $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid and for all $a, b, c \in A$,

$$a \cdot b \leq c \text{ iff } b \leq a\backslash c \text{ iff } a \leq c/b.$$

We refer to the last property as *residuation.* As usual, we write xy for $x \cdot y$. We also write $x\backslash y/z$ for $x\backslash(y/z)$ and $(x\backslash y)/z$, since the latter two are equal in every residuated lattice.

An *FL-algebra* is a residuated lattice $\mathbf{A}$ with a distinguished element $0 \in A$. The element 0 is used to define negations: ${\sim}x = x\backslash 0$, $-x = 0/x$. As usual, we will write $a \leq b$ instead of $a = a \wedge b$ and ab instead of $a \cdot b$. FL-algebras are noncommutative in general, so the two implications $\backslash, /$ are in general different. When $a\backslash b = b/a$ (or the distinction between them is irrelevant), we write $a \to b$; likewise we write $\neg x$ if ${\sim}x = -x$. If we assume that the lattice reduct is bounded, we may also include the constants $\top$ and $\bot$ in the language. However, we do not assume boundedness in general, since it excludes interesting algebras, such as lattice-ordered groups.

We denote by FL the variety of FL-algebras. Given a set E of equations, we denote by $\mathsf{FL_E}$ the subvariety of FL that consists of algebras satisfying E. Typical equations are:

$$(e)\quad xy \leq yx \qquad (w)\quad 0 \leq x \leq 1 \qquad (c)\quad xx \leq x$$

In $\mathsf{FL_e}$, we always have $x\backslash y = y/x = x \to y$. In $\mathsf{FL_w}$, we have $1 = \top$ and $0 = \bot$. In $\mathsf{FL_{ewc}}$ we have $x \cdot y = x \wedge y$. Hence $\mathsf{FL_{ewc}}$ is term equivalent to the variety HA of Heyting algebras.

For a class K of algebras, we denote by K_{SI} the class of subdirectly irreducible algebras in K.

We denote by and, or and $\Longrightarrow$ the Boolean conjunction, disjunction and implication of the underlying first-order language, respectively. A *clause* is a universal first-order formula of the form ($n \geq m \geq 0$)

$$t_1 \leq u_1 \ \mathsf{and}\ \ldots\ \mathsf{and}\ t_m \leq u_m \Longrightarrow t_{m+1} \leq u_{m+1}\ \mathsf{or}\ \ldots\ \mathsf{or}\ t_n \leq u_n. \qquad (q)$$

$t_1 \leq u_1, \ldots, t_m \leq u_m$ are called *premises* and $t_{m+1} \leq u_{m+1}, \ldots, t_n \leq u_n$ *conclusions*.

Definition 2.1. A clause (q) is said to be *structural* if t_i is a product of variables (including the empty product 1) and u_i is either 0 or a variable x for every $1 \leq i \leq n$. A clause with only one conclusion is called a *quasiequation*.

An FL-algebra $\mathbf{A}$ *satisfies* (q) if for every valuation f into $\mathbf{A}$, $f(t_i) \leq f(u_i)$ for all $1 \leq i \leq m$ implies $f(t_j) \leq f(u_j)$ for some $m+1 \leq j \leq n$.

Given a class K of algebras, we say that two clauses (q_1) and (q_2) are *equivalent* in K if they are satisfied by the same algebras in K.

A *completion* of an FL-algebra $\mathbf{A}$ is a pair $(\mathbf{B}, i)$ where $\mathbf{B}$ is a complete FL-algebra and $i : \mathbf{A} \longrightarrow \mathbf{B}$ is an embedding. We will often identify $\mathbf{A}$ with $i(\mathbf{A}) \subseteq \mathbf{B}$. The variety of FL-algebras admits MacNeille completions, and moreover the latter have an abstract characterization [2, 13]:

Theorem 2.2. *Every FL-algebra $\mathbf{A}$ has a completion $\overline{\mathbf{A}} \in \mathsf{FL}$, unique up to isomorphism, such that $\mathbf{A}$ is both meet dense and join dense in $\overline{\mathbf{A}}$. Namely, every element a of $\overline{A}$ can be written as*

$$a = \bigvee X = \bigwedge Y \quad \textit{for some } X, Y \subseteq A.$$

$\overline{\mathbf{A}}$ is called the *MacNeille completion* of $\mathbf{A}$. It is a *regular* completion, namely, the embedding of $\mathbf{A}$ in $\overline{\mathbf{A}}$ preserves all existing joins and meets.

2.2. Substructural Hierarchy. Proof-theoretic considerations (see [1, 4]) suggest the classification of the operations of FL (including $\top, \bot$) into two groups (*polarities*):

- *positive* operations: $1, \bot, \cdot, \vee$
- *negative* operations: $\top, 0, \wedge, \backslash, /$

This classification makes algebraic sense as natural identities hold among operations of the same polarity. For positive operations, we have:

$$\begin{aligned} x \cdot 1 &= x \\ x \vee \bot &= x \\ x \cdot \bot &= \bot \\ x \cdot (y \vee z) &= (x \cdot y) \vee (x \cdot z) \end{aligned}$$

For negative operations, we have:

$$\begin{aligned} x \wedge \top &= x \\ 1 \to x &= x \\ x \to \top &= \top \\ x \to (y \wedge z) &= (x \to y) \wedge (x \to y) \\ (x \vee y) \to z &= (x \to z) \wedge (y \to z) \end{aligned}$$

Here $\to$ stands for both $\backslash$ and $/$ (uniformly in the same equation). Notice that the second and fifth equations above involve the positive operations 1 and $\vee$; this is because we assume that polarity is reversed on the left hand side of $\to$, a fact that is related to the order-reversing nature of the operation in its first argument.

The *substructural hierarchy* $(\mathcal{P}_n, \mathcal{N}_n)$, introduced in [4, 5] to classify FL terms, is based on alternations of positive and negative layers.

Definition 2.3. For each $n \geq 0$, the sets $\mathcal{P}_n, \mathcal{N}_n$ of FL terms are defined as follows:

(0) $\mathcal{P}_0 = \mathcal{N}_0$ is the set of variables.
(P1) $1, \bot$ and all terms in $\mathcal{N}_n$ belong to $\mathcal{P}_{n+1}$.
(P2) If $t, u \in \mathcal{P}_{n+1}$, then $t \vee u, t \cdot u \in \mathcal{P}_{n+1}$.
(N1) $0, \top$ and all terms in $\mathcal{P}_n$ belong to $\mathcal{N}_{n+1}$.
(N2) If $t, u \in \mathcal{N}_{n+1}$, then $t \wedge u \in \mathcal{N}_{n+1}$.
(N3) If $t \in \mathcal{P}_{n+1}$ and $u \in \mathcal{N}_{n+1}$, then $t\backslash u, u/t \in \mathcal{N}_{n+1}$.

Namely $\mathcal{P}_{n+1}$ is the set generated from $\mathcal{N}_n$ by means of finite joins (including the empty join $\bot$) and products (including the empty product 1), and $\mathcal{N}_{n+1}$ is generated by $\mathcal{P}_n \cup \{0\}$ by means of finite meets (including the empty meet $\top$) and divisions with denominators from $\mathcal{P}_{n+1}$.

We say that $1 \leq t$ belongs to $\mathcal{P}_n$ ($\mathcal{N}_n$, resp.) if t does. Also, if $s \neq 1$, we say that $s \leq t$ belongs to $\mathcal{P}_n$ ($\mathcal{N}_n$, resp.) if $s\backslash t$ does. We also extend the classification to equations $s = t$, by choosing the join in the classification poset of the classes of $s \leq t$ and $t \leq s$.

For every n, we have $\mathcal{P}_n \cup \mathcal{N}_n \subseteq \mathcal{P}_{n+1}$ and $\mathcal{P}_n \cup \mathcal{N}_n \subseteq \mathcal{N}_{n+1}$. Hence the hierarchy can be depicted as in Figure 1.

We recall from [5] that terms in each class admit the following normal forms:

Lemma 2.4.

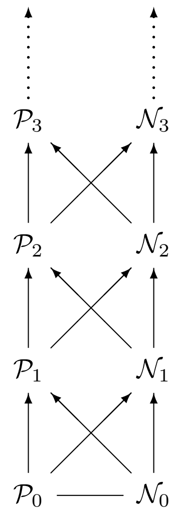

Figure 1. The Substructural Hierarchy

(P) *Every term in* $\mathcal{P}_{n+1}$ *is equivalent in* FL *to* $\bot$ *or to* $u_1 \vee \cdots \vee u_m$*, where each* u_i *is a product of terms in* $\mathcal{N}_n$*.*

(N) *Every term in* $\mathcal{N}_{n+1}$ *is equivalent in* FL *to a finite meet of terms of the form* $l\backslash u/r$*, where* u *is either* 0 *or a term in* $\mathcal{P}_n$*, and* l, r *are products of terms in* $\mathcal{N}_n$*.*

Proof. We will prove the lemma by simultaneous induction of the two statements.

Let $t \in \mathcal{P}_{n+1}$. Statement (P) is clear for $t = \bot$. The case $t = 1$ is a special case for $m = 1$ and u_1 the empty product. If (P) holds for $t, u \in \mathcal{P}_{n+1}$, then it clearly holds for $t \vee u$. For $t \cdot u$, we use the fact that multiplication distributes over joins.

Let $t \in \mathcal{N}_{n+1}$. Statement (N) is clear for $t = \top$. For $t = 0$ we take $l = r = 1$ and $u = 0$. If (N) holds for $t, u \in \mathcal{N}_{n+1}$, then it clearly holds for $t \wedge u$. If $t \in \mathcal{P}_{n+1}$ and $u \in \mathcal{N}_{n+1}$, we know that $t = t_1 \vee \cdots \vee t_m$, for t_i a product of terms in $\mathcal{N}_n$, where $m = 0$ yields the empty join $t = \bot$. We have $t\backslash u = (t_1 \vee \cdots \vee t_m)\backslash u = (t_1\backslash u) \wedge \cdots \wedge (t_m\backslash u)$. Moreover, by the induction hypothesis, for all $j \in \{1, \ldots, m\}$, $t_j\backslash u = t_j\backslash(\bigwedge_{1\leq i\leq k} l_i\backslash u_i/r_i) = \bigwedge_{1\leq i\leq k} t_j\backslash(l_i\backslash u_i/r_i) = \bigwedge_{1\leq i\leq k}(l_i t_j)\backslash u_i/r_i$; the empty meet $\top$ is obtained for $k = 0$. $\square$

Corollary 2.5 (Normal form of $\mathcal{N}_2$ equations)**.** *Every equation in* $\mathcal{N}_2$ *is equivalent in* FL *to a finite conjunction of equations of the form* $t_1 \cdots t_m \leq u$ *where* $u = 0$ *or* $u = u_1 \vee \cdots \vee u_k$ *with each* u_i *a product of variables. Furthermore, each* t_i *is of the form* $\bigwedge_{1\leq j\leq n} l_j\backslash v_j/r_j$ *(or* $\bigwedge_{1\leq j\leq n} l_j \to v_j$*, for* $\mathsf{FL_e}$*), where* $v_j = 0$ *or a variable, and* l_j *and* r_j *are products of variables.*

Class	Equation	Name
$\mathcal{N}_2$	$xy \leq yx$	commutativity
	$x \leq 1,\ 0 \leq x$	integrality
	$x \leq xx$	contraction
	$xx \leq x$	expansion
	$x^n \leq x^m$	knotted axioms
	$1 \leq \neg(x \wedge \neg x)$	no-contradiction
$\mathcal{P}_2$	$1 \leq x \vee \neg x$	excluded middle
	$1 \leq (x \to y) \vee (y \to x)$	prelinearity
$\mathcal{P}_3$	$1 \leq \neg x \vee \neg\neg x$	weak excluded middle
	$1 \leq x \vee \neg x^{n-1}$	n-excluded middle
	$1 \leq \neg(xy) \vee (x \wedge y \to xy)$	weak nilpotent minimum
	$1 \leq x(x \backslash 1)$	ℓ-group
	$1 \leq \bigvee_{i=0}^{k}(p_0 \wedge \cdots \wedge p_{i-1} \to p_i)$	(Bdk), $k \geq 1$
	$1 \leq p_0 \vee \cdots \vee (p_0 \wedge \cdots \wedge p_{k-1} \to p_k)$	(Bck), $k \geq 1$
	$1 \leq x \vee (x \to 1)$	semiconicity
$\mathcal{N}_3$	$x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$	distributivity
	$x \to xy \leq y$	cancellativity
	$(x \wedge y) \leq x(x \to y)$	divisibility

Figure 2. The zoo of equations

Some examples of equations classified into the hierarchy are found in Figure 2 (where $\to$ can be any of $\backslash$ and $/$ and x^n stands for $x \cdot \ldots \cdot x$, n times).

Remark 2.6. In [14, 15], Zakharyaschev proved that all superintuitionistic logics are definable by his *canonical formulas*. Since each canonical formula can be expressed by an equation in $\mathcal{N}_3$ (if one carefully chooses between $\cdot$ and $\wedge$ for conjunction), it follows that all subvarieties of HA are definable by equations in $\mathcal{N}_3$. Namely, the substructural hierarchy *collapses* beyond level $\mathcal{N}_3$ over HA.

We say that a class of algebras *admits MacNeille completions*, if for every algebra in the class, its MacNeille completion is also in the class. We say that a class *admits completions*, if every algebra in the class embeds in some complete algebra also in the class.

Theorem 2.7 ([5])**.** *Let E be a set of $\mathcal{N}_2$ equations. The following are equivalent:*

- $\mathsf{FL_E}$ *admits completions.*
- $\mathsf{FL_E}$ *admits MacNeille completions.*
- *E is equivalent to a set of acyclic quasiequations (cf. Definition 5.5) in* FL.

Moreover, if integrality ($x \leq 1$) belongs to E, $\mathsf{FL_E}$ is always closed under MacNeille completions.

The last statement means that all $\mathcal{N}_2$ equations are preserved by MacNeille completions when applied to algebras in $\mathsf{FL_w}$.

3. From $\mathcal{P}_3$ equations to analytic clauses

In general, proving that a certain equation is preserved by MacNeille completions is not an easy task, especially if the equation has a complicated form. In this section we show how to transform $\mathcal{P}_3$ equations, in the commutative and integral case, into clauses of a simple syntactic form. These clauses—called *analytic*—lend themselves to a uniform proof of preservation under MacNeille completions. Our transformation, inspired by the proof theoretic results in [4], is done in two steps: (1) we first "unfold" $\mathcal{P}_3$ equations into structural clauses and (2) structural clauses are transformed (completed, in the sense of Knuth-Bendix) into analytic clauses.

We begin with two easy lemmas. The first recalls some standard properties of $\mathsf{FL_{ew}}$.

Lemma 3.1. *Let* $\mathbf{A} \in \mathsf{FL_{ew}}$ *and let* t, u *be FL-terms.*

(1) $\mathbf{A} \models 1 \leq t \cdot u$ *if and only if* $\mathbf{A} \models 1 \leq t$ *and* $\mathbf{A} \models 1 \leq u$.
(2) $\mathbf{A} \models 1 \leq t$ or $1 \leq u$ *implies* $\mathbf{A} \models 1 \leq t \vee u$.
(3) *If* $\mathbf{A}$ *is subdirectly irreducible,* $\mathbf{A} \models 1 \leq t \vee u$ *implies* $\mathbf{A} \models 1 \leq t$ or $1 \leq u$.

Proof. First note that all inequalities can be replaced by equalities in the presence of integrality. (1) follows from order preservation of multiplication and (2) is a trivial observation about join. (3) follows from results in [12], or in [6]. $\square$

The next lemma, which is the key for our transformation procedure, is based on a simple observation: every equation $t \leq u$ is equivalent to a quasiequation $u \leq x \Longrightarrow t \leq x$, and also to $x \leq t \Longrightarrow x \leq u$, where x is a fresh variable not occurring in t, u. This is a very particular case of Yoneda's lemma in category theory. Below, $\vec{\varepsilon}_1$ (resp., $\vec{\varepsilon}_2$) stands for a Boolean conjunction (resp., disjunction) of equations.

Lemma 3.2. *The following clauses are equivalent in* FL*, where* x *is a fresh variable.*

(1) $\vec{\varepsilon}_1 \Longrightarrow \vec{\varepsilon}_2$ or $ltr \leq u$.
(2) $\vec{\varepsilon}_1$ and $u \leq x \Longrightarrow \vec{\varepsilon}_2$ or $ltr \leq x$.
(3) $\vec{\varepsilon}_1$ and $x \leq t \Longrightarrow \vec{\varepsilon}_2$ or $lxr \leq u$.

Proof. We prove the equivalence between (1) and (2). The other case is similar. (1) follows from (2) by instantiating x with u in (2). For the converse, assume that the premises of (2) hold. By (1) and $\vec{\varepsilon}_1$ we get $\vec{\varepsilon}_2$ or $ltr \leq u$. The conclusion of (2) follows from this and $u \leq x$. $\square$

Theorem 3.3. *Every equation ε in $\mathcal{P}_3$ is equivalent in $(\mathsf{FL_{ew}})_{SI}$ to a finite set R of structural clauses. More precisely, ε implies R in $(\mathsf{FL_{ew}})_{SI}$, while R implies ε in $\mathsf{FL_{ew}}$.*

Proof. By Lemma 2.4, every equation in $\mathcal{P}_3$ is equivalent in FL to an equation of the form $1 \leq \bigvee \prod s_{ij}$ with $s_{ij} \in \mathcal{N}_2$; here $\bigvee$ denotes a finite join and $\prod$ a finite product. In view of Lemma 3.1, this is equivalent in $(\mathsf{FL_{ew}})_{SI}$ to a conjunction of disjunctions of equations $1 \leq s_{ij}$, namely to a finite set of clauses of the form

$$1 \leq s_1 \text{ or } 1 \leq s_2 \text{ or } \dots \text{ or } 1 \leq s_m,$$

where each $1 \leq s_i$ is in $\mathcal{N}_2$. By Corollary 2.5, each $1 \leq s_i$ is equivalent to an equation of the form $t_{i1} \cdots t_{in_i} \leq u_i$ with $u_i = 0$ or $u_i = u_{i1} \vee \cdots \vee u_{ik_i}$ and each u_{ij} is a product of variables. Furthermore each t_{ij} $(j = 1, \dots, n_i)$ is of the form $\bigwedge_{1 \leq p \leq q_{ij}} l_p \to v_p$, where $v_p = 0$ or a variable, and l_p are products of variables. In other words the clause becomes

$$t_{11} \cdots t_{1n_1} \leq u_1 \text{ or } \dots \text{ or } t_{m1} \cdots t_{mn_m} \leq u_m,$$

By repeatedly applying Lemma 3.2, it further becomes

$$u_1 \leq z_1 \text{ and } \dots \text{ and } u_m \leq z_m \Longrightarrow t_{11} \cdots t_{1n_1} \leq z_1 \text{ or } \dots \text{ or } t_{m1} \cdots t_{mn_m} \leq z_m.$$

which we write in more compact form

$$\underset{1 \leq i \leq m}{\text{AND}} \ u_i \leq z_i \Longrightarrow \underset{1 \leq i \leq m}{\text{OR}} \ t_{i1} \cdots t_{in_i} \leq z_i,$$

By further applications of Lemma 3.2, we get

$$\underset{1 \leq i \leq m}{\text{AND}} \ u_i \leq z_i \text{ and } \underset{\substack{1 \leq i \leq m \\ 1 \leq j \leq n_i}}{\text{AND}} \ x_{ij} \leq t_{ij} \Longrightarrow \underset{1 \leq i \leq m}{\text{OR}} \ x_{i1} \cdots x_{in_i} \leq z_i,$$

Given the form of u_i and t_{ij}, the equations $u_i \leq z_i$ and $x_{ij} \leq t_{ij}$ become

$$\bigvee_{1 \leq j \leq k_i} u_{ij} \leq z_i \qquad \text{and} \qquad x_{ij} \leq \bigwedge_{1 \leq p \leq q_{ij}} l_p \to v_p,$$

respectively (we adopt here the convention that for $k_i = 0$, $u_{i1} \vee \cdots \vee u_{ik_i}$ denotes $\perp$). The clause then becomes

$$\underset{\substack{1 \leq i \leq m \\ 1 \leq j \leq k_i}}{\text{AND}} \ u_{ij} \leq z_i \text{ and } \underset{\substack{1 \leq i \leq m \\ 1 \leq j \leq n_i \\ 1 \leq p \leq q_{ij}}}{\text{AND}} \ l_p x_{ij} \leq v_p \Longrightarrow \underset{1 \leq i \leq m}{\text{OR}} \ x_{i1} \cdots x_{in_i} \leq z_i,$$

which is structural.

The last claim of the theorem (R implies ε in $\mathsf{FL_{ew}}$) can be verified by tracing back the transformation steps above. Notice that we do not need to use Lemma 3.1 (3) for this direction. □

We introduce below a procedure to simplify structural clauses in the presence of integrality, that is to transform them into equivalent analytic clauses.

Definition 3.4. Given a structural clause

$$t_1 \le u_1 \text{ and } \dots \text{ and } t_m \le u_m \Longrightarrow t_{m+1} \le u_{m+1} \text{ or } \dots \text{ or } t_n \le u_n, \qquad (q)$$

we call the variables occurring in $t_{m+1}, \dots, t_n$ *left* variables, and those in $u_{m+1}, \dots, u_n$ *right* variables. The set of left (resp., right) variables is denoted by $L(q)$ (resp., $R(q)$). (q) is said to be *analytic* if it satisfies the following conditions:

Separation: $L(q)$ and $R(q)$ are disjoint.

Linearity: Any variable $x \in L(q) \cup R(q)$ occurs exactly once in t_{m+1}, u_{m+1}, ..., t_n, u_n.

Inclusion: $t_1, \dots, t_m$ are made of variables in $L(q)$ (here repetition is allowed), while $u_1, \dots, u_m$ are made of variables in $R(q)$ (and 0).

Remark 3.5. Analytic clauses correspond to the proof theoretic notion of analytic structural rules, see [4]. These are structural rules in a generalization of Gentzen sequent calculus, which behave well with respect to cut-elimination.

Theorem 3.6. *Every structural clause is equivalent in* $\mathsf{FL_w}$ *to an analytic one.*

Proof. Given a structural clause

$$t_1 \le u_1 \text{ and } \dots \text{ and } t_m \le u_m \Longrightarrow t_{m+1} \le u_{m+1} \text{ or } \dots \text{ or } t_n \le u_n \qquad (q)$$

the transformation into an analytic clause equivalent to (q) in $\mathsf{FL_w}$ proceeds in two steps:

1. Restructuring. For each $i \in \{m+1, \dots n\}$, assume that t_i is $y_1 \cdots y_p$. Let $x_0, x_1, \dots, x_p$ be distinct fresh variables. Depending on whether u_i is 0 or a variable, we transform (q) into either

$$S \text{ and } x_1 \le y_1 \text{ and } \dots \text{ and } x_p \le y_p \Longrightarrow S' \text{ or } x_1 \dots x_p \le 0 \quad \text{or} \qquad (q_1)$$

$$S \text{ and } x_1 \le y_1 \text{ and } \dots \text{ and } x_p \le y_p \text{ and } u_i \le x_0 \Longrightarrow S' \text{ or } x_1 \dots x_p \le x_0 \qquad (q_2)$$

where S denotes the set of premises of (q) and S' denotes the conclusion of (q) without $t_i \le u_i$ (i.e. $t_{m+1} \le u_{m+1}$ or ... or $t_{i-1} \le u_{i-1}$ or $t_{i+1} \le u_{i+1}$ or ... or $t_n \le u_n$).

The equivalence of (q_1) (or (q_2)) to (q) in FL follows by Lemma 3.2. We apply this procedure for all $i \in \{m+1, \dots n\}$.

2. Cutting. Let (q') be the clause obtained after step 1 (restructuring). (q') satisfies the properties of separation and linearity of Definition 3.4. An analytic clause equivalent to (q') in $\mathsf{FL_w}$ is obtained by suitably removing all the *redundant variables* from its premises, i.e., variables other than $L(q') \cup R(q')$. This is done as follows: Let z be any redundant variable. If z appears only on right-hand sides (RHS) of premises we simply remove all such premises, say $s_1 \le z, \dots, s_k \le z$ from (q'). It is easy to see that the resulting clause is equivalent to (q') in FL. Indeed observe that all premises $s_i \le z$ in (q') hold by instantiating z with $\bigvee s_i$, and that the instantiation does not affect the other premises and conclusion. Hence (q') implies the new clause. The other direction is trivial.

If z appears only on left-hand sides (LHS) of premises of (q'), we again remove all such premises, say $l_1 \cdot z \cdot r_1 \leq v_1, \ldots, l_k \cdot z \cdot r_k \leq v_k$ from (q'). We argue similarly as in the previous case, instantiating z with $\bigwedge_{1 \leq i \leq k} l_i \backslash v_i / r_i$.

Otherwise, z appears both on RHS and LHS of premises of (q'). Let S_r and S_l be the sets of premises of (q') which involve z on RHS and LHS, respectively. Namely, S_r consists of the premises of the form $s \leq z$, and S_l of the form $t(z, \ldots, z) \leq v$, where the occurrences of z in t are all indicated.

Because of integrality, we can assume that S_r and S_l are disjoint. Indeed, if an equation belongs to both S_r and S_l, then it is of the form $t(z, \ldots, z) \leq z$, which can be safely removed as it follows from integrality. We replace $S_r \cup S_l$ with a new set S_{cut} of premises, which consists of all equations of the form

$$t(s_1, \ldots, s_k) \leq v, \quad \text{where } t(z, \ldots, z) \leq v \in S_l \text{ and } s_1 \leq z, \ldots, s_k \leq z \in S_r.$$

The resulting clause implies (q'), in view of transitivity. To show the converse, assume the premises of the new one. By instantiating $z = \bigvee s = \bigvee\{s : s \leq z \in S_r\}$, all premises in S_r hold and all premises $t(\bigvee s, \ldots, \bigvee s) \leq v$ in S_l follow from S_{cut}. Hence (q') yields the conclusion.

The claim follows by applying the cutting step for each redundant variable. □

Example 3.7. Prelinearity $1 \leq (x \to y) \vee (y \to x)$ is equivalent in $(\mathsf{FL_{ew}})_{SI}$ to the structural clause $x \leq y$ or $y \leq x$. By applying the completion procedure in the proof of Theorem 3.6 we can transform the latter into the (equivalent) analytic clause

$$z \leq x \text{ and } w \leq y \Longrightarrow w \leq x \text{ or } z \leq y. \qquad (com)$$

Example 3.8. The smallest non-trivial, non-Boolean subvariety of Heyting algebras is generated by the 3-element Heyting algebra; it is axiomatized, relative to Heyting algebras, by the equation (Bc2) $1 \leq x \vee (x \to y) \vee [(x \wedge y) \to z]$ (also known as (sm)) together with prelinearity. J. Harding proved [9] that this variety admits regular completions; i.e., every algebra in the variety embeds in a complete algebra in the variety via a map that preserves existing (arbitrary) meets and joins. (Bc2) $\in \mathcal{P}_3$ (cf. Figure 2) and it is equivalent in $(\mathsf{FL_{ew}})_{SI}$ to the structural clause $1 \leq x$ or $x \leq y$ or $x \wedge y \leq z$. By applying the completion procedure in the proof of Theorem 3.6, we can transform the latter into the (equivalent) analytic clause

$$u \leq x \text{ and } w \leq x \text{ and } w \leq y \Longrightarrow 1 \leq x \text{ or } u \leq y \text{ or } w \leq z. \qquad \text{(Clause-Bc2)}$$

4. Preservation by MacNeille completions

We provide a uniform proof of the preservation of analytic clauses by MacNeille completions. Together with the results in the previous section this implies that each subvariety of $\mathsf{FL_{ew}}$ axiomatized by $\mathcal{P}_3$ equations admits completions.

Theorem 4.1. *If an analytic clause is satisfied by an FL-algebra* $\mathbf{A}$*, then it is also satisfied by its MacNeille completion* $\overline{\mathbf{A}}$.

Before proving Theorem 4.1, let us explain the idea by means of an example.

Example 4.2. Consider the analytic clause (com) $z \leq x$ and $w \leq y \Longrightarrow w \leq x$ or $z \leq y$, which is equivalent in $(\mathsf{FL_{ew}})_{SI}$ to prelinearity (see Example 3.7). Assume that $\mathbf{A} \in \mathsf{FL}$ satisfies (com); we prove that $\overline{\mathbf{A}}$ satisfies it too. By Theorem 2.2, any element of $\overline{\mathbf{A}}$ can be written as both a join and a meet of elements of $\mathbf{A}$. Hence it is enough to show

$$\bigvee Z \leq \bigwedge X \text{ and } \bigvee W \leq \bigwedge Y \Longrightarrow \bigvee W \leq \bigwedge X \text{ or } \bigvee Z \leq \bigwedge Y$$

for $X, Y, Z, W \subseteq A$. Assume by way of contradiction that $\bigvee Z \leq \bigwedge X$, $\bigvee W \leq \bigwedge Y$, $\bigvee W \not\leq \bigwedge X$ and $\bigvee Z \not\leq \bigwedge Y$. From the latter two, we can choose $x \in X$, $y \in Y$, $z \in Z$, $w \in W$ such that $w \not\leq x$ and $z \not\leq y$. On the other hand, the former two imply $z \leq x$ and $w \leq y$. But then (com) yields either $w \leq x$ or $z \leq y$ — a contradiction.

Following the example above, the idea in the proof is to represent the variables that appear on the left-hand side of inequalities as joins and those on the right-hand side as meets of elements of $\mathbf{A}$. The separation property guarantees that this can be done for each analytic clause.

Proof. Assume that $\mathbf{A} \in \mathsf{FL}$ satisfies an analytic clause

$$t_1 \leq u_1 \text{ and } \dots \text{ and } t_m \leq u_m \Longrightarrow t_{m+1} \leq u_{m+1} \text{ or } \dots \text{ or } t_n \leq u_n. \qquad (q)$$

Let $L(q) \cup R(q) = \{x_1, \dots, x_p\}$. To prove that $\overline{\mathbf{A}}$ satisfies (q), let f be a valuation into $\overline{A}$ defined on $L(q) \cup R(q)$. By Theorem 2.2 and the separation condition, we may assume that there are $X_1, \dots, X_p \subseteq A$ such that $f(x_k) = \bigvee X_k$ if $x_k \in L(q)$ and $f(x_k) = \bigwedge X_k$ if $x_k \in R(q)$.

Assume by way of contradiction that $f(t_i) \leq f(u_i)$ for every $1 \leq i \leq m$ and $f(t_j) \not\leq f(u_j)$ for every $m+1 \leq j \leq n$. From the latter, we can choose one a_k from each X_k that make all the conclusions false. More precisely, there exist $a_k \in X_k$ such that if we define a valuation g into A by $g(x_k) = a_k$, then $g(t_j) \not\leq g(u_j)$ for all $m+1 \leq j \leq n$. Notice that it is the linearity condition that allows us to pick up exactly one a_j from each X_j making all conclusions false, so that the valuation g is well defined.

The valuation g satisfies all the premises. Indeed, assume that $t_i = x_{i_1} \dots x_{i_l}$ and $u_i = x_{i_0}$ for $1 \leq i \leq m$. By the inclusion condition, $x_{i_1}, \dots, x_{i_l} \in L(q)$

and $x_{i_0} \in R(q)$. Hence we have

$$\begin{aligned} g(t_i) &= a_{i_1} \dots a_{i_l} \\ &\leq (\bigvee X_{i_1}) \dots (\bigvee X_{i_l}) = f(t_i) \\ &\leq f(u_i) = \bigwedge X_{i_0} \\ &\leq a_{i_0} = g(u_i). \end{aligned}$$

(The case $u_i = 0$ is similar; we then have $f(u_i) = g(u_i) = 0$.)

Since $\mathbf{A}$ satisfies (q), we must have $g(t_j) \leq g(u_j)$ for some $m+1 \leq j \leq n$. That is a contradiction. □

By combining Theorems 3.3, 3.6 and 4.1, we obtain:

Theorem 4.3. *If a $\mathcal{P}_3$ equation is satisfied by a subdirectly irreducible algebra $\mathbf{A}$ in $\mathsf{FL_{ew}}$, then it is also satisfied by its MacNeille completion $\overline{\mathbf{A}}$.*

Corollary 4.4. *If a subvariety of $\mathsf{FL_{ew}}$ is axiomatized by $\mathcal{P}_3$ equations, it admits completions.*

Proof. Any algebra $\mathbf{A}$ in such a variety $\mathcal{V}$ embeds into a product $\prod_{i \in I} \mathbf{A}_i$ of subdirectly irreducible algebras $\mathbf{A}_i$ in $\mathcal{V}$. By the above theorem, their MacNeille completions $\overline{\mathbf{A}_i}$ also belong to $\mathcal{V}$. Hence we have a completion

$$\mathbf{A} \hookrightarrow \prod_{i \in I} \mathbf{A}_i \hookrightarrow \prod_{i \in I} \overline{\mathbf{A}}_i \in \mathcal{V}.$$

□

Remark 4.5. The completion in Corollary 4.4 is not necessarily regular because the subdirect representation does not preserve existing joins.

5. The noncommutative and nonintegral cases

In extending Theorem 4.3 to FL_{SI} we meet two obstacles: (a) In the absence of commutativity and integrality Lemma 3.1, which is crucial for transforming any equation into equivalent structural clauses, does not hold. (b) The cutting step to complete structural clauses into analytic ones (Theorem 3.6) does not go through without integrality. In this section we show that these two obstacles can be overcome by suitably modifying the class of equations we deal with and by imposing a further syntactic condition on the structural clauses to be completed. As discussed in Section 6 the proposed restrictions are unavoidable.

To simulate the effects of commutativity and integrality without actually assuming them, we consider the notion of *iterated conjugate*.

A *conjugate* of a term t is either $\lambda_u(t) = (u \backslash tu) \wedge 1$ or $\rho_u(t) = (ut/u) \wedge 1$ for some term u. Conjugates indeed allow us to simulate integrality and commutativity, as

$$\lambda_u(t) \leq 1, \qquad \rho_u(t) \leq 1, \qquad u\lambda_u(t) \leq tu, \qquad \rho_u(t)u \leq ut.$$

Since one application of commutativity/integrality corresponds to the insertion of a conjugate, repeated applications of these properties require iterated uses of conjugates. An *iterated conjugate* of t is a term of the form $\gamma_{u_1}\cdots\gamma_{u_n}(t)$, where each γ_{u_i} is either λ_{u_i} or ρ_{u_i}.

Notation: We add an auxiliary unary operation $\flat$ to the language of FL. The intended meaning is that $\flat t$ corresponds to the set of all iterated conjugates of t.

Definition 5.1. For each $n \geq 0$, we denote by $\mathcal{P}'_{n+1}$ the set of terms generated from $\{\flat t : t \in \mathcal{N}_n\}$ by finite joins and products. More precisely:

- If $t \in \mathcal{N}_n$, then $\flat t \in \mathcal{P}'_{n+1}$.
- $1, \bot \in \mathcal{P}'_{n+1}$.
- If $t, u \in \mathcal{P}'_{n+1}$, then $t \vee u, t \cdot u \in \mathcal{P}'_{n+1}$.

We say that an equation $1 \leq t$ belongs to $\mathcal{P}'_n$ if t does.

To each $t \in \mathcal{P}'_{n+1}$, we assign the set $FL(t)$ of FL terms as follows:

$$\begin{aligned}
FL(\flat t) &= \{t' : t' \text{ is an iterated conjugate of } t\}\\
FL(t \cdot u) &= \{t' \cdot u' : t' \in FL(t), u' \in FL(u)\}\\
FL(t \vee u) &= \{t' \vee u' : t' \in FL(t), u' \in FL(u)\}\\
FL(\bot) &= \{\bot\}\\
FL(1) &= \{1\}
\end{aligned}$$

Given an FL-algebra $\mathbf{A}$ and an equation $1 \leq t$ with $t \in \mathcal{P}'_{n+1}$, we write $\mathbf{A} \models 1 \leq t$ if $\mathbf{A} \models 1 \leq t'$ for every $t' \in FL(t)$.

Below are some examples of $\mathcal{P}'_3$ equations:

$$1 \leq \flat(x \to y) \vee \flat(y \to x), \quad 1 \leq \flat\neg(x \cdot y) \vee \flat(x \wedge y \to x \cdot y), \quad 1 \leq \flat x \vee \flat y \vee \flat(xy \backslash yx).$$

The use of iterated conjugates allows us to prove an analogue of Lemma 3.1, thus overcoming obstacle (a), mentioned at the very beginning of the section.

Lemma 5.2. *Let $\mathbf{A}$ be an FL-algebra and let t, u be FL-terms.*

(1) $\mathbf{A} \models 1 \leq \flat t \cdot \flat u$ *if and only if* $\mathbf{A} \models 1 \leq t$ *and* $\mathbf{A} \models 1 \leq u$.
(2) $\mathbf{A} \models 1 \leq t$ or $1 \leq u$ *implies* $\mathbf{A} \models 1 \leq \flat t \vee \flat u$.
(3) *If $\mathbf{A}$ is subdirectly irreducible,* $\mathbf{A} \models 1 \leq \flat t \vee \flat u$ *implies* $\mathbf{A} \models 1 \leq t$ or $1 \leq u$.

Proof. First note that: (*) $1 \leq t$ iff $1 \leq \gamma(t)$, for every iterated conjugate γ. Indeed, for the forward direction take $\gamma(t) = \lambda_1(t) = 1\backslash(t \cdot 1) \wedge 1 = t \wedge 1$, to get $1 \leq t \wedge 1$, i.e., $1 \leq t$. For the converse, if $1 \leq t$, then for all a, $1 \leq t \leq a\backslash(ta)$, hence $1 \leq a\backslash(ta) \wedge 1 = \lambda_a(t)$, and likewise $1 \leq \rho_a(t)$. By repeated applications of this principle, we get $1 \leq \gamma(t)$, for every iterated conjugate γ.

As in the proof of Lemma 3.1, (1) and (2) follow easily, by also using (*). In detail for (1), $\mathbf{A}$ satisfies $1 \leq \flat t \cdot \flat u$ iff for all iterated conjugates γ and γ', $\mathbf{A}$ satisfies $1 \leq \gamma(t) \cdot \gamma'(u)$. As each conjugate is a negative element (i.e., less or equal to 1), this is equivalent to: $1 \leq \gamma(t)$ and $1 \leq \gamma'(u)$, for all conjugates.

Now (1) follows from (*). Finally, (3) follows in this generality directly from results in [6]. □

Having obtained this, it is now straightforward to modify the proof of Theorem 3.3.

Theorem 5.3. *Every equation in $\mathcal{P}_3'$ is equivalent in FL_{SI} to a finite set of structural clauses.*

Remark 5.4. In presence of commutativity, we can identify $\flat t$ with $t \wedge 1$. In this case $\mathcal{P}_3'$ can be seen as a subclass of $\mathcal{P}_3$ (a proper subclass, as shown in Section 6).

We now consider the second obstacle (b) in extending Theorem 4.3 to FL_{SI}: in the absence of integrality, the cutting step in the proof of Theorem 3.6 does not go through. In analogy to [5], we use the following side condition on equations and structural clauses.

Definition 5.5. Given a structural clause (q), we build its *dependency graph* $D(q)$ in the following way:

- The vertices of $D(q)$ are the variables occurring in the premises (we do not distinguish occurrences).
- There is a directed edge $x \longrightarrow y$ in $D(q)$ if and only if there is a premise of the form $lxr \leq y$.

(q) is *acyclic* if the graph $D(q)$ is acyclic (i.e., it has no directed cycles or loops).

A $\mathcal{P}_3'$-equation ε is said to be *acyclic* if so are all the structural clauses obtained by applying to ε the procedure described in the proof of Theorem 3.3.

Example 5.6. Consider the $\mathcal{N}_2$ equation $x\backslash x \leq x/x$. By applying the procedure in the proof of Theorem 3.3 we obtain the equivalent (in FL) quasiequation

$$xy \leq x \Longrightarrow yx \leq x. \tag{we}$$

(we) is not acyclic, since we have a loop at the vertex x in $D(we)$.

Theorem 5.7. *Every acyclic clause is equivalent to an analytic one in* FL.

Proof. We proceed as in the proof of Theorem 3.6. Integrality was used there in the cutting step to assume that the sets S_r and S_l of premises which involve a redundant variable z on the RHS and LHS were disjoint. The acyclicity condition trivially guarantees that this is the case. □

We have done all the modifications needed to obtain a suitable generalization of Theorem 4.3.

Theorem 5.8. *If a $\mathcal{P}_3'$ equation (resp. acyclic $\mathcal{P}_3'$ equation) is satisfied by a subdirectly irreducible algebra $\mathbf{A}$ in $\mathsf{FL_w}$ (resp. FL), then it is also satisfied by its MacNeille completion $\overline{\mathbf{A}}$.*

Corollary 5.9. *If a subvariety of* $\mathsf{FL_w}$ *(resp.* FL*) is axiomatized by* $\mathcal{P}_3'$ *equations (resp. acyclic* $\mathcal{P}_3'$ *equations), then it admits completions.*

We note again that $\mathcal{P}_3'$ can be seen as a subclass of $\mathcal{P}_3$ over $\mathsf{FL_e}$, since we can identify $\flat t$ with $t \wedge 1$.

6. Perspectives

We have shown that a wide class of equations is preserved by MacNeille completions when applied to subdirectly irreducible FL-algebras, therefore a wide class of varieties of FL-algebras admits completions. On the other hand, there exist varieties that do *not* admit completions. These varieties provide some negative answers to the natural questions below:

Does Theorem 5.8 hold for all $\mathcal{P}_3$ *equations?:* The variety LG of lattice-ordered groups does not admit completions, simply because it does not contain any nontrivial bounded algebra. LG is axiomatized by $1 \leq x(x\backslash 1)$ over FL. The equation is in $\mathcal{P}_3$, but not in $\mathcal{P}_3'$.

Can the acyclicity condition be dropped?: In [5] we proved that the subvariety of FL defined by $x\backslash x \leq x/x$ does not admit any completion (and therefore, by Theorem 2.7 it is not equivalent to any analytic clause). The above equation belongs to $\mathcal{N}_2 \subseteq \mathcal{P}_3$ but it is not acyclic (see Example 5.6).

Beyond $\mathcal{P}_3$*?:* The subvariety of $\mathsf{FL_{ew}}$ defined by the cancellativity axiom $x \to xy \leq y \in \mathcal{N}_3$ does not admit completions for the same reason as LG. Also notably, the varieties of MV, BL, GMV and GBL algebras, all axiomatized by certain $\mathcal{N}_3$ equations over $\mathsf{FL_w}$, do not admit completions either (see [11]). For instance, the variety of BL algebras is axiomatized by prelinearity and the $\mathcal{N}_3$ equation of divisibility over $\mathsf{FL_{ew}}$ (see Figure 2).

These facts indicate that our results are more or less optimal.

As noted in Remark 4.5, our completions of varieties are not necessarily regular, since they use the subdirect representation, which does not preserve existing joins. On the other hand, cut-elimination for the proof-theoretic counterpart of $\mathcal{P}_3$ equations [4] suggests the definition of a new kind of completion. In our subsequent work we will introduce such completions, called *hyper-MacNeille* completions, which do preserve all $\mathcal{P}_3$ equations not only for subdirectly irreducible FL_{ew}-algebras, but also for arbitrary FL_{ew}-algebras. Circumventing subdirect representation, these completions turn out to be regular, hence any subvariety of $\mathsf{FL_{ew}}$ axiomatized by $\mathcal{P}_3$ equations admit regular completions.

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Agata Ciabattoni

Department of Computer Languages, Vienna University of Technology, Favoritenstrasse 9-11, 1040 Wien, Austria
e-mail: agata@logic.at

Nikolaos Galatos

Department of Mathematics, University of Denver, 2360 S. Gaylord St., Denver, CO 80208, USA
e-mail: ngalatos@du.edu

Kazushige Terui

Research Institute for Mathematical Sciences, Kyoto University, Kitashirakawa Oiwakecho, Sakyo-ku, Kyoto 606-8502, Japan
e-mail: terui@kurims.kyoto-u.ac.jp