Modules over Quantaloids: Applications to the Isomorphism Problem in Algebraic Logic and π-institutions

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Abstract

We solve the isomorphism problem in the context of abstract algebraic logic and of π-institutions, namely the problem of when the notions of syntactic and semantic equivalence among logics coincide. The problem is solved in the general setting of categories of modules over quantaloids. We prove that these categories are strongly complete, strongly cocomplete, and $(\mathrm{Epi}, \mathrm{Mono})$-structured. We prove a duality property, a characterization of monos in virtue of Yoneda Lemma, and as a consequence of this and of the duality, a characterization of epis. We introduce closure operators and closure systems on modules over quantaloids, and its associated morphisms. We show that, up to isomorphism, epis are morphisms associated with closure operators, and as a consequence that $\mathrm{Epi} = \mathrm{RegEpi}$, and by duality $\mathrm{Mono} = \mathrm{RegMono}$. This is fundamental in the proof of the strong amalgamation property.

The notions of (semi-)interpretability and (semi-)representability are introduced and studied. We introduce cyclic modules, and provide a characterization for cyclic projective modules as those having a g-variable. From this we obtain that categories of modules over (small) quantaloids have enough injectives and projectives.

Finally, we explain how every π-institution induces a module over a quantaloid, and thus the theory of modules over quantaloids can be considered as an abstraction of the theory of π-institutions.

1 Introduction

In order to study the property of algebraizability for sentential logics, and the equivalence between deductive systems in general, Blok and Jónsson introduced the notion of *equivalence* between *structural closure operators* on a set X acted on by a monoid M, or an *M-set* (see [BJ06]). As usual, given a monoid $(M, \cdot, 1)$, an M-set consists of a set X and a monoid action $\star : M \times X \to X$, where $1 \star x = x$ and $a \star (b \star x) = (a \cdot b) \star x$, for all $a, b \in M$ and $x \in X$. While the use of closure operators to encode entailment relations is very well known, the action of the monoid is introduced to formalize the notion of structurality, that is, "entailments are preserved by uniform substitutions," a property usually required for logics.

Given an M-set $\langle X, \cdot\rangle$, a closure operator C on X is *structural* on $\langle X, \cdot\rangle$ if and only if it satisfies the following property: for every $\sigma \in M$, and every $\Gamma \subseteq X$, $\sigma \cdot C\Gamma \subseteq C(\sigma \cdot \Gamma)$, where $\sigma \cdot \Gamma = \{\sigma \cdot \varphi : \varphi \in \Gamma\}$. This can be shortly written as follows:

$$\forall \sigma \in M, \quad \sigma C \leqslant C\sigma. \tag{Str}$$

This is known as the *structurality property* for C, since it takes the following form, when expressed in terms of $\vdash_C$, the closure relation on X associated with the closure operator C (defined by $\varphi \in C\Gamma$ iff $\Gamma \vdash_C \varphi$): for every $\Gamma \subseteq X$, every $\varphi \in X$, and every $\sigma \in M$,

$$\Gamma \vdash_C \varphi \;\Rightarrow\; \sigma \cdot \Gamma \vdash_C \sigma \cdot \varphi.$$

For ever $\sigma \in M$, a unary operation $C\sigma$ on $\mathbf{Cl}(C)$, the lattice of *theories* or *closed sets* of C, is defined in the following way: $C\sigma(\Gamma) = C(\sigma \cdot \Gamma)$. The *expanded lattice of theories* of a structural closure operator C is defined as the structure $\langle \mathbf{Cl}(C), (C\sigma)_{\sigma \in M} \rangle$.

In their approximation, Blok and Jónsson define two structural closure operators on two M-set s to be equivalent if their expanded lattices of theories are isomorphic. Later, they prove that under certain hypotheses (the existence of basis), this is equivalent to the existence of conservative and mutually inverse interpretations, which is the original idea of equivalence between deductive systems emerging from the work of Blok and Pigozzi. This equivalence between the lattice-theoretic property of having isomorphic expanded lattices of theories, and the semantic property of being mutually interpretable is known by the name of the *Isomorphism Theorem*. And the problem of determining in which situations there exists an Isomorphism Theorem is called the *Isomorphism Problem*.

The first Isomorphism Theorem was proved by Blok and Pigozzi in [BP89] for algebraizable sentential logics, and later it was obtained for k-dimensional deductive systems by them in [BP92] and for Gentzen systems by Rebagliato and Verdú in [RV95]. But there is not a general Isomorphism Theorem for structural closure operators on M-set s, as there are counterexamples for that (see [GFér]).

In turn, Voutsadakis studied in [Vou03] the notion of equivalence of π-institutions at different levels (quasi-equivalence and deductive equivalence) and identified term π-institutions, for which a certain kind of Isomorphism Theorem also holds. The notion of π-institution was introduced by Fiadeiro and Sernadas in their article [FS88] and can be viewed as a generalization of deductive systems allowing multiple sorts. They constitute a very wide categorical framework embracing sentential logics, Gentzen systems, etc., as they include structural closure operators on M-set s as a particular case. Therefore, a general Isomorphism Theorem for π-institutions is not possible (see [GF06]).

Sufficient conditions for the existence of an Isomorphism Theorem were provided in [GFér] and [GF06] for structural closure operators on M-set s (and graduated M-set s), and π-institutions that encompass all the previous known cases. The first complete solution of the Isomorphism Problem was found for closure operators on modules over residated complete lattices, or *quantales* (see [GT09]). In this article, the modules providing an Isomorphism Theorem are identified as the projective modules. In particular, cyclic projective modules are characterized in several ways, from which the Isomorphism Theorem for k-deductive systems follows, and also for Gentzen systems, using that coproducts of projectives are projective. The Isomorphism Problem for π-institutions remained open.

In this article we present a solution for the Isomorphism Problem in the general framework of closure operators on modules over quantaloids, and as a particular case for π-institutions. In order to do that, the theory of modules over quantaloids and of closure operators on them is developed in a categorical way. This yields a very rich theory with many nice properties: We prove that there exists a duality in the categories of modules over quantaloids, that they are strongly complete and strongly cocomplete, that they are (Epi, Mono)-structured, that they have enough injectives and projectives, and that they satisfy the strong amalgamation property, among others. Many of these results are generalizations of the same results obtained by Solovyov for categories of modules over quantales (see [Sol08]). We characterize monos and epis in the categories of modules over quantaloids, and furthermore prove that every epi is induced by a closure operator on its domain.

We also study the notions of closure system on a module over a quantaloid, and prove that they are exactly the submodules of the dual module, and that the standard correspondence between closure operators and closure systems on a set extends to a natural isomorphism. We introduce the notions of (semi-)interpretability and (semi-)representability of one closure operator into another and study their relationships. We prove that the set of closure operators that are interpretable by a given morphism τ is a principal filter of the lattice of closure operators on its domain. As a consequence, we obtain that every extension of an interpretable closure operator is also interpretable by the same morphism. One instantiation of this result is the well-known fact that if a sentential logic has an algebraic semantics, then every extension of it also has an algebraic semantics and with the same defining equations. This is the contents of Theorem 2.15 of [BR03].

One of the main results of the paper is Theorem 9.10, where the modules with the property that every representation of a closure operator on them into another closure operator is induced are nicely characterized as the projective ones. This is the key result for Theorem 11.3, which establishes that every equivalence between two closure operators on projective modules is induced by mutually inverse interpretations. That is the general solution for the Isomorphism Problem in the setting of modules over quantaloids.

In the last section we explain in detail how every π-institution induces a closure operator on a module over a quantaloid, and every translation between π-institutions induces a morphism in the fibered category of all modules over quantaloids. Thus, we show how the theory of closure operators on modules over quantaloids is a generalization of the theory of interpretations and representations of π-institutions.

2 Modules coming from M-set s

We start our study by reviewing modules over a quantale and how every M-set induces a module over a quantale. Then, we provide a characterization of cyclic and projective modules coming from M-set s as those whose M-set has a generalized variable, a condition very close to the property of having a variable (see [GFér]), which is proved to be not necessary (see Theorem 2.10 and Example 2.11). This will be generalized to modules over quantaloids, in Section 10 (see Theorem 10.11).

Recall that a *quantale* $\langle A, \bigvee, \cdot, 1\rangle$ is a join-complete lattice and a monoid such that multiplication distributes on both sides over (arbitrary) joins. Note that quantales are definitionally equivalent to *complete residuated lattices*, namely algebras $\langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$, with a monoid and a complete lattice reduct such that $x \cdot y \leq z \Leftrightarrow x \leq z/y \Leftrightarrow y \leq x\backslash z$, for all $x, y, z \in A$; however, homomorphisms differ, so quantales and complete residuated lattices give rise to different categories. Also, recall from [GT09] that, given a quantale $\mathcal{A} = \langle \mathbf{A}, \cdot, 1\rangle$, an $\mathcal{A}$-*module* is a pair $\langle \mathbf{R}, *\rangle$, where $\mathbf{R}$ is a complete lattice and $* : \mathbf{A} \times \mathbf{R} \to \mathbf{R}$ is a biresiduated map satisfying that for every $x \in R$ and $a, b \in A$:

(M1) $1 * x = x$,
(M2) $a * (b * x) = (ab) * x$.

Recall that for partially ordered sets $\mathbf{P}$ and $\mathbf{Q}$ a map $\tau : P \to Q$ is called *residuated* if there exists a map $\tau^+ : Q \to P$ such that $\tau(p) \leq q \Leftrightarrow p \leq \tau^+(q)$, for all $p \in P$ and $q \in Q$. For complete lattices $\mathbf{P}$ and $\mathbf{Q}$, $\tau : P \to Q$ is residuated iff it preserves (arbitrary) joins. For partially ordered sets $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$, a map $\tau : P \times Q \to R$ is called *biresiduated* if the sections $\tau_p : y \mapsto \tau(p, y)$ and $\tau_q : x \mapsto \tau(x, q)$ are residuated for all $p \in P$ and $q \in Q$. Note that the last condition in the definition of a residuated lattice states that multiplication is biresiduated.

Following [GT09], there is a standard way of defining a quantale $\mathcal{A}_M$ from a monoid M, lifting all the operations to the powersets: let $\mathbf{A}$ be the complete lattice with universe $\mathcal{P}(M)$ of subsets of M ordered by inclusion, and for every $a, b \in A$, $a \cdot b = \{\sigma\sigma' : \sigma \in a,\ \sigma' \in b\}$. Then, $\mathcal{A}_M = \langle \mathbf{A}, \cdot, \{1_M\}\rangle$ is a quantale. Analogously, given an M-set $\langle X, \cdot\rangle$ the $\mathcal{A}_M$-module associated with it is defined as follows: take $R = \mathcal{P}(X)$ and for every $a \in A$ and $x \in R$, $a * x = \{\sigma \cdot \varphi : \sigma \in a,\ \varphi \in x\}$. Thus, $\mathbb{R} = \big\langle \langle \mathcal{P}(X), \subseteq\rangle, *\big\rangle$ is an $\mathcal{A}_M$-module.

Recall from [GFér] the following notion: an element $q \in X$ is a *variable* for an M-set $\langle X, \cdot\rangle$ if there exists a family $\kappa = \{\kappa_\varphi : \varphi \in X\} \subseteq M$, called a *uniform* or *coherent family of substitutions*, such that

(i) for every $\varphi \in X$, $\kappa_\varphi \cdot q = \varphi$,
(ii) for every $\sigma \in M$, and every $\varphi \in X$, $\sigma\kappa_\varphi = \kappa_{\sigma\varphi}$.

We note that the notion of a variable is a proper generalization of the notion of a basis (in the sense of Blok and Jónsson): if an M-set has a basis B, then every $q \in B$ is a variable; also there are examples of M-sets with variables that do not have any basis. Using terminology in the references that will be introduced later in the paper, we mention that it was proved in [GFér] that every representation of a structural closure operator on an M-set with a variable into another is induced, whence these M-set s are well-behaved (see Theorem 34 of [GFér], where the more general graded-version is established). In [GT09] the $\mathcal{A}$-modules with the property that every structural representation of a closure operator on them into another is induced are characterized as the O-projectives, where $O = \mathrm{Onto}(\mathcal{A}\text{-}Mod)$ is the class of all morphisms in $\mathcal{A}\text{-}Mod$ that are ontos.

Recall that, if $\mathcal{E}$ is a family of morphisms of some category, then an object P of this category is *$\mathcal{E}$-projective* if for every $\varepsilon : R \to S$ in $\mathcal{E}$, any morphism $f : P \to S$ can be extended to a morphism $\overline{f} : P \to R$ such that $\varepsilon\overline{f} = f$:

$$\begin{tikzcd} & R \arrow[d, "\varepsilon"] \\ P \arrow[ur, dashed, "\exists\overline{f}"] \arrow[r, "f"'] & S \end{tikzcd}$$

An object P is *projective* if it is Epi-projective, where Epi is the class of all epimorphisms in this category.

Theorem 2.1 (Galatos-Tsinakis)**.** *Let $\mathcal{A}$ be a quantale and $O = \mathrm{Onto}(\mathcal{A}\text{-}Mod)$ the class of all $\mathcal{A}$-morphisms that are ontos. An $\mathcal{A}$-module $\mathbb{R}$ is cyclic and O-projective if and only if there exists $u \in A$ such that $u = u^2$ and $\mathbb{R} \cong \mathbb{A} \cdot u$.* □

We will prove later (see Corollary 6.3) a characterization for $\mathrm{Onto}(\mathcal{Q}\text{-}Mod)$ for an arbitrary quantaloid $\mathcal{Q}$, that in particular implies that for every quantale $\mathcal{A}$, $\mathrm{Onto}(\mathcal{A}\text{-}Mod) = \mathrm{Epi}(\mathcal{A}\text{-}Mod)$, and then the prefix "O-" can be removed in Theorem 2.1. As a consequence of these results we can establish the following proposition:

Proposition 2.2. *Let M be a monoid, $\langle X, \cdot\rangle$ an M-set, $\mathbb{R}$ the $\mathcal{A}_M$-module associated with $\langle X, \cdot\rangle$. If $\langle X, \cdot\rangle$ has a variable then $\mathbb{R}$ is a cyclic and projective $\mathcal{A}_M$-module.*

Proof. Let $q \in X$ be a variable and $\kappa = \{\kappa_\varphi : \varphi \in X\}$ a uniform family of substitutions for q. If we set $u = \{\kappa_q\} \in A$, then it is immediate that $A \cdot u = \mathcal{P}(\kappa)$. Note that $u^2 = \{\kappa_q\kappa_q\} = u$, since $\kappa_q\kappa_q = \kappa_{\kappa_q \cdot q} = \kappa_q$.

The map $f : R \to \mathcal{P}(\kappa) = A \cdot u$ defined by $fx = \{\kappa_\varphi : \varphi \in x\}$ is bijective with inverse $f^{-1} : \mathcal{P}(\kappa) \to R$ defined by $f^{-1}(a) = \{\sigma \cdot q : \sigma \in a\}$. Furthermore, the map $f : \mathbf{R} \to \langle \mathcal{P}(\kappa), \subseteq\rangle$ is a residuated map, since $f(\bigcup x_i) = \bigcup f(x_i)$. Moreover, $f(a * x) = f\big(\{\sigma\cdot\varphi : \sigma \in a,\ \varphi \in x\}\big) = \{\kappa_{\sigma\cdot\varphi} : \sigma \in a,\ \varphi \in x\} = \{\sigma \cdot \kappa_\varphi : \sigma \in a,\ \varphi \in x\} = a \cdot \{\kappa_\varphi : \varphi \in x\} = a \cdot f(x)$.

Thus, $f : \mathbb{R} \to \mathbb{A}_M \cdot u$ is an $\mathcal{A}_M$-isomorphism, and in virtue of Theorem 2.1 and Corollary 6.3, $\mathbb{R}$ is cyclic and projective. □

In view of the preceding result, the following are natural questions: Under which conditions can a reciprocal of the preceding proposition be proved? Does every M-set giving rise to a cyclic and projective $\mathcal{A}_M$-module have a variable? We will answer these questions, but in order to do that, we first need to further analyze the notion of a variable, as well as of cyclic projective. The following is also a specialization of a theorem in [GT09].

Proposition 2.3 (Galatos-Tsinakis)**.** *If $\langle X, \cdot\rangle$ is an M-set and $\mathbb{R}$ is the associated $\mathcal{A}_M$-module, then $\mathbb{R}$ is cyclic and projective if and only if there exists $v \subseteq X$ and $u \subseteq M$ such that:*

1. *$u * v = v$,*
2. *for every $x \subseteq X$, there exists $v_x \subseteq M$, such that $v_x * v = x$,*
3. *for every $a \subseteq M$, $(a * v/v)u = au$.* □

Remark 2.4. Note that condition 3. can be rewritten in the following terms: for every $\pi \in M$, and $a \subseteq M$,

$$\{\pi\} * v \subseteq a * v \ \Rightarrow\ \{\pi\}u \subseteq au. \tag{1}$$

It follows that, for every $a, b \subseteq M$,

$$b * v = a * v \ \Rightarrow\ bu = au. \tag{2}$$

The next proposition shows that having a variable is pretty close to having a cyclic and projective associated module.

Proposition 2.5. *If $\langle X, \cdot \rangle$ is an M-set, $p \in X$ and $\kappa = \{\kappa_\varphi : \varphi \in X\} \subseteq M$, then p is a variable for $\langle X, \cdot \rangle$ with uniform family of substitutions κ if and only if*

(i') for every $\varphi \in X$, $\kappa_\varphi \kappa_p = \kappa_\varphi$,
(ii') for every $\varphi \in X$, $\kappa_\varphi \cdot p = \varphi$,
(iii') for every $\pi, \sigma \in M$, $\pi \cdot p = \sigma \cdot p \Rightarrow \pi\kappa_p = \sigma\kappa_p$.

Proof. If p is a variable for $\langle X, \cdot \rangle$ with uniform family of substitutions κ, then (ii') is trivial. By (ii) and (i) we obtain:

$$\kappa_\varphi \kappa_p = \kappa_{\kappa_\varphi \cdot p} = \kappa_\varphi,$$

whence we have (i'). Finally, let π, σ be elements in M such that $\pi \cdot p = \sigma \cdot p$. Therefore, by (ii), $\pi\kappa_p = \kappa_{\pi \cdot p} = \kappa_{\sigma \cdot p} = \sigma\kappa_p$.

Suppose now that p and κ satisfy (i'), (ii') and (iii'), and let us prove (ii), since (i) is trivial by (ii'). For every $\sigma \in M$, we have that

$$\sigma\kappa_\varphi \cdot p = \sigma \cdot \varphi = \kappa_{\sigma\cdot\varphi} \cdot p,$$

and therefore, by (iii'), $\sigma\kappa_\varphi\kappa_p = \kappa_{\sigma\cdot\varphi}\kappa_p$. And then, in virtue of (i'), $\sigma\kappa_\varphi = \kappa_{\sigma\cdot\varphi}$. □

Remark 2.6. Note that (ii') and (iii') implies that, in particular $\kappa_p^2 = \kappa_p$, since $\kappa_p \cdot p = p = id \cdot p$.

In fact, Condition (i') in the preceding proposition is somehow redundant. It cannot be derived from the other two, as the next example shows, but given p and κ satisfying (ii') and (iii'), we can modify κ in order to obtain another κ' satisfying all the three properties.

Example 2.7. Let $X = \{p, q, 0\}$ be a set with three elements, and $\kappa_p, \kappa_q, \kappa_0 : X \to X$ the maps determined by:

$$\begin{array}{ccc} p \overset{\kappa_p}{\longmapsto} p & p \overset{\kappa_q}{\longmapsto} q & p \overset{\kappa_0}{\longmapsto} 0 \\ q \longmapsto 0 & q \longmapsto q & q \longmapsto 0 \\ 0 \longmapsto 0 & 0 \longmapsto 0 & 0 \longmapsto 0 \end{array}$$

Let M the monoid generated by these three maps, and $\langle X, \cdot \rangle$ the M-set determined by $\sigma \cdot \varphi = \sigma\varphi$, for $\sigma \in M$ and $\varphi \in X$. It is straightforward to prove that, if id is the identity map on X and $\tau = \kappa_q\kappa_p$, then $M = \{id, \kappa_p, \kappa_q, \kappa_0, \tau\}$ and the multiplication table is:

	κ_p	κ_q	κ_0	τ
κ_p	κ_p	κ_0	κ_0	κ_0
κ_q	τ	κ_q	κ_0	τ
κ_0	κ_0	κ_0	κ_0	κ_0
τ	τ	κ_0	κ_0	κ_0

By definition, $\tau p = q$ and $\tau q = \tau 0 = 0$. For $\pi, \sigma \in M$, $\pi \neq \sigma$, we have that $\pi \cdot p = \sigma \cdot p$ if and only if $\{\pi, \sigma\} = \{\kappa_p, id\}$ or $\{\pi, \sigma\} = \{\kappa_q, \tau\}$. And then, since $\kappa_p\kappa_p = \kappa_p = id\,\kappa_p$ and $\kappa_q\kappa_p = \tau = \tau\kappa_p$, we have that p and $\kappa = \{\kappa_p, \kappa_q, \kappa_0\}$ satisfy (ii') and (iii'). But, $\kappa_q\kappa_p = \tau \neq \kappa_q$, and then they do not satisfy (i').

Proposition 2.8. *If $\langle X, \cdot \rangle$ is an M-set, and $p \in X$ and $\kappa = \{\kappa_\varphi : \varphi \in X\} \subseteq M$ satisfy (ii') and (iii'), then p is a variable for $\langle X, \cdot \rangle$ with uniform family of substitutions κ' defined by $\kappa'_\varphi = \kappa_\varphi\kappa_p$.*

Proof. By Remark 2.6, $\kappa_p^2 = \kappa_p$, or in other words $\kappa'_p = \kappa_p$. Therefore, condition (iii') for p and κ' is trivially satisfied.

Now, for every $\varphi \in X$, $\kappa'_\varphi \cdot p = \kappa_\varphi\kappa_p \cdot p = \kappa_\varphi \cdot p = \varphi$. Moreover, $\kappa'_\varphi\kappa'_p = \kappa_\varphi\kappa_p\kappa_p = \kappa_\varphi\kappa_p^2 = \kappa_\varphi\kappa_p = \kappa'_\varphi$, which concludes the proof. □

Comparing the conditions in Proposition 2.3 and in Proposition 2.5, we observe that 2. and 3. are very similar to (ii') and (iii'), respectively, where κ_p plays the role of u and p stands for v. Indeed, if p is a variable for $\langle X, \cdot\rangle$ with uniform family of substitutions κ, and we take $v = \{p\}$ and $u = \{\kappa_p\}$, then the tree conditions of Proposition 2.3 are satisfied, which constitutes another proof that the associated $\mathcal{A}_M$-module $\mathbb{R}$ is cyclic and projective.

But, if we are under the hypotheses of Proposition 2.3 and we search for a variable of $\langle X, \cdot\rangle$ it seems that we have to reduce v and u to singletons. Indeed, this inspires the following definition, and we say that p is a *generalized variable* when v is just the singleton $v = \{p\}$.

Definition 2.9. A *generalized variable* or *g-variable* for an M-set $\langle X, \cdot\rangle$ is an element $p \in X$ such that there exist $u \subseteq M$, satisfying the following three conditions:

1'. $u * \{p\} = \{p\}$,
2'. for every $\varphi \in X$, there exists $v_\varphi \in M$, such that $v_\varphi \cdot p = \varphi$,
3'. for every $\pi, \sigma \in M$, if $\pi \cdot p = \sigma \cdot p$, then $\{\pi\}u = \{\sigma\}u$.

We can now prove the following characterization.

Theorem 2.10. *Let $\langle X, \cdot\rangle$ be an M-set and $\mathbb{R}$ the associated $\mathcal{A}_M$-module. $\mathbb{R}$ is cyclic and projective if and only if $\langle X, \cdot\rangle$ has a g-variable.*

Proof. Suppose that $\langle X, \cdot\rangle$ has a g-variable p and that $u \subseteq M$ and $v_\varphi \in M$ for every $\varphi \in X$ satisfy conditions 1'., 2'., and 3'. Let $v = \{p\}$, and for every $x \subseteq X$, let $v_x = \{v_\varphi : \varphi \in x\}$. It is easy to check that conditions 1. and 2. are satisfied in virtue of 1'. and 2'. In order to prove 3., suppose that $\pi \in M$ and $a \subseteq M$ are such that $\{\pi\} * \{p\} \subseteq a * \{p\}$, that is, $\pi \cdot p \in a * \{p\}$, and let us show that $\{\pi\}u \subseteq au$. Since $\pi \cdot p \in a * \{p\}$, there exists $\sigma \in a$ such that $\pi \cdot p = \sigma \cdot p$, and by 3'., $\{\pi\} * u = \{\sigma\}u \subseteq au$.

Suppose now that $\mathbb{R}$ is cyclic and projective and let $v \subseteq X$, $u \subseteq M$ and for every $x \subseteq X$, $v_x \subseteq M$ be elements satisfying conditions 1., 2., and 3.

Let $p \in v$, for every $\varphi \in X$, let $v_\varphi \in v_{\{\varphi\}}$, and $u' = \{v_p\}u$. We will show that 1'., 2'., and 3'. are satisfied by u', p and v_φ. Note that, for every $\varphi \in X$, $\{v_\varphi\} * \{p\} \subseteq v_{\{\varphi\}} * v = \{\varphi\}$, and therefore $v_\varphi \cdot p = \varphi$. Then, we have that,

$$u' * \{p\} \subseteq u' * v = (\{v_p\}u) * v = \{v_p\} * (u * v) = \{v_p\} * v \subseteq v_{\{p\}} * v = \{p\},$$

whence we obtain that $u' * \{p\} = u' * v = \{p\}$.

Suppose now that $\pi, \sigma \in M$ are such that $\pi \cdot p = \sigma \cdot p$. Hence,

$$\begin{aligned}(\{\pi\}u') * v &= \{\pi\} * (u' * v) = \{\pi\} * \{p\} = \{\pi \cdot p\} = \{\sigma\} * \{p\} = \{\sigma\} * (u' * v) \\ &= (\{\sigma\}u') * v.\end{aligned}$$

Since we are assuming 3., in particular by (2) we obtain that $\{\pi\}u'u = \{\sigma\}u'u$. Let us prove that $u'u = u'$, and we will be done. Since $v = u * v$, we have $\{v_p\} * v = (\{v_p\}u) * v = u' * v$, and then by (2),

$$u' = \{v_p\}u = u'u.$$

□

Hence, under the hypotheses of Proposition 2.3, we can always reduce v to a singleton, but we cannot do the same with u, as it is shown by the next example.

Example 2.11. Let $X = \{p, s_1, s_2\}$ be a set with three elements, and $\kappa_1, \kappa_2, e_1, e_2 : X \to X$ the maps determined by:

$$\begin{array}{cccc} p \overset{\kappa_1}{\longmapsto} p & p \overset{\kappa_2}{\longmapsto} p & p \overset{e_1}{\longmapsto} s_1 & p \overset{e_2}{\longmapsto} s_2 \\ s_1 \longmapsto s_1 & s_1 \longmapsto s_2 & s_1 \longmapsto s_1 & s_1 \longmapsto s_2 \\ s_2 \longmapsto s_1 & s_2 \longmapsto s_2 & s_2 \longmapsto s_1 & s_2 \longmapsto s_2 \end{array}$$

Let M the monoid generated by these three maps, and $\langle X, \cdot\rangle$ the M-set determined by $\sigma \cdot \varphi = \sigma\varphi$, for $\sigma \in M$ and $\varphi \in X$. It is straightforward to prove that, if id is the identity map on X then $M = \{id, \kappa_1, \kappa_2, e_1, e_2\}$.

It is easy to check that, by taking $u = \{\kappa_1, \kappa_2\}$, $v_p = \kappa_1$, $v_{s_1} = e_1$ and $v_{s_2} = e_2$, conditions 1'. and 2'. are satisfied. Moreover, for every $\pi, \sigma \in M$, $\pi \neq \sigma$, we have that $\pi \cdot p = \sigma \cdot p$ if and only if $\pi, \sigma \in \{\kappa_1, \kappa_2, id\}$. But, we have the equations

$$u = \{id\}u = \{\kappa_1\}u = \{\kappa_2\}u,$$

which prove that condition 3'. is also satisfied. That is, $\langle X, \cdot \rangle$ has a g-variable, p.

Nevertheless, this M-set does not have a variable. The only possible candidate for a variable is p, and therefore the only candidates for κ_p are id, κ_1 and κ_2. If we take $\kappa_p = id$, then it does not satisfy condition (iii'), because $\kappa_1\, id = \kappa_1 \neq \kappa_2 = \kappa_2\, id$. Analogously, the inequalities $\kappa_1\kappa_1 = \kappa_1 \neq \kappa_2 = \kappa_2\kappa_1$ and $\kappa_2\kappa_2 = \kappa_2 \neq \kappa_1 = \kappa_1\kappa_2$ show that neither κ_1 nor κ_2 can be taken to be κ_p. Hence, p cannot be a variable.

3 The monoidal structure of the category $\mathcal{S}\ell$

Let us denote by $\mathcal{S}\ell$ the category of join-complete lattices. That is, the objects of $\mathcal{S}\ell$ are join-complete lattices and its arrows are maps preserving arbitrary joins. It is well known that join-complete lattices are complete lattices, and that the maps between complete lattices preserving arbitrary joins are exactly the residuated maps.

Given $\mathbf{R}, \mathbf{S}, \mathbf{T} \in \mathcal{S}\ell$, a map $f : R \times S \to T$ is *biresiduated* (with respect to $\mathbf{R}$, $\mathbf{S}$ and $\mathbf{T}$) if it is residuated in each variable, that is, for every $a \in R$, and every $b \in S$, the maps $f(a, \_) : \mathbf{S} \to \mathbf{T}$ and $f(\_, b) : \mathbf{R} \to \mathbf{T}$ are residuated. Since all the considered lattices are complete, being biresiduated is equivalent to preserving arbitrary joins in each variable separately. Note that, since $\mathcal{S}\ell$ has products, if $\mathbf{R}, \mathbf{S} \in \mathcal{S}\ell$ then $\mathbf{R} \times \mathbf{S} \in \mathcal{S}\ell$, but a map f biresiduated with respect to $\mathbf{R}$, $\mathbf{S}$, and $\mathbf{T}$, is not in general a residuated map from $\mathbf{R} \times \mathbf{S}$ to $\mathbf{T}$. That is, biresiduated maps are not, in general, morphisms of $\mathcal{S}\ell$. Nevertheless, we will also use the arrow notation $\mathbf{R} \times \mathbf{S} \to \mathbf{T}$ to denote biresiduated maps, and hope no confusion arises from this practice.

In the same manner as bilinear maps can be "encoded" by linear maps via the tensor product of vectorial spaces, biresiduated maps can also be "encoded" by residuated maps, as we explain in what follows.

As it is mentioned in [JT84], for every $\mathbf{R}, \mathbf{S} \in \mathcal{S}\ell$ there exist a complete lattice $\mathbf{R} \otimes \mathbf{S}$ and a biresiduated map $\mathbf{R} \times \mathbf{S} \to \mathbf{R} \otimes \mathbf{S}$ that is universal among all the biresiduated maps. That is, every biresiduated map $f : \mathbf{R} \times \mathbf{S} \to \mathbf{T}$ factorizes through it via a unique residuated map:

$$\begin{tikzcd}
\mathbf{R} \times \mathbf{S} \arrow[r] \arrow[dr, "f"'] & \mathbf{R} \otimes \mathbf{S} \arrow[d, dashed, "\exists!\, \overline{f}"] \\
& \mathbf{T}
\end{tikzcd}$$

The *tensor product* of $\mathbf{R}$ and $\mathbf{S}$ is defined as the codomain $\mathbf{R} \otimes \mathbf{S}$ of this universal biresiduated map. It is therefore unique up to isomorphism. The image of a pair $\langle x, y \rangle \in \mathbf{R} \times \mathbf{S}$ in the tensor product is denoted by $x \otimes y$. The tensor product can be constructed by the standard methods as a quotient of $\mathbf{R} \times \mathbf{S}$ by the congruence generated by $\langle \bigvee x_i, y \rangle \equiv \bigvee \langle x_i, y \rangle$ and $\langle x, \bigvee y_i \rangle \equiv \bigvee \langle x, y_i \rangle$, for arbitrary families $\{x_i : i \in I\} \subseteq R$ and $\{y_i : i \in I\} \subseteq S$, and elements $x \in R$, $y \in S$. The interested reader is referred to the general study of bimorphisms[1] developed in [BN76] for more details about this kind of constructions.

On the other direction, since the composition gh of a residuated map g with a biresiduated map h is biresiduated, there exists a bijection between biresiduated maps $\mathbf{R} \times \mathbf{S} \to \mathbf{T}$ and residuated maps $\mathbf{R} \otimes \mathbf{S} \to \mathbf{T}$, and in this sense we say that we "encode" biresiduated maps by residuated maps.

The map assigning to every pair of complete lattices their tensor product extends to a bifunctor $\otimes : \mathcal{S}\ell \times \mathcal{S}\ell \to \mathcal{S}\ell$ in the following way: if $\langle f, g \rangle : \langle \mathbf{R}, \mathbf{S} \rangle \to \langle \mathbf{R}', \mathbf{S}' \rangle$ is a morphism

[1] There is another notion of bimorphism: that of a morphism in a category that is simultaneously epi and mono (see [AHS06]). We do not use the term *bimorphism* with this sense.

in $\mathcal{S}\ell \times \mathcal{S}\ell$, then consider the residuated map $f \times g : \mathbf{R} \times \mathbf{S} \to \mathbf{R}' \times \mathbf{S}'$. The composition of $f \times g$ with the universal map $\mathbf{R}' \times \mathbf{S}' \to \mathbf{R}' \otimes \mathbf{S}'$ is biresiduated, and therefore, by the universal property of $\mathbf{R} \otimes \mathbf{S}$, the following diagram can be completed:

$$\begin{tikzcd} \mathbf{R} \times \mathbf{S} \arrow[r] \arrow[d, "f \times g"'] \arrow[dr] & \mathbf{R} \otimes \mathbf{S} \arrow[d, dashed, "f \otimes g"] \\ \mathbf{R}' \times \mathbf{S}' \arrow[r] & \mathbf{R}' \otimes \mathbf{S}' \end{tikzcd} \qquad \begin{tikzcd} \langle x, y \rangle \arrow[r, mapsto] \arrow[d, mapsto] & x \otimes y \arrow[d, mapsto, dashed] \\ \langle fx, fy \rangle \arrow[r, mapsto] & (f \otimes g)(x \otimes y) = fx \otimes gy \end{tikzcd}$$

It is straightforward to prove that there exist natural isomorphisms $\lambda : \mathbf{2} \otimes \mathbf{R} \to \mathbf{R}$, $\rho : \mathbf{R} \otimes \mathbf{2} \to \mathbf{R}$, and $\alpha : (\mathbf{R} \otimes \mathbf{S}) \otimes \mathbf{T} \to \mathbf{R} \otimes (\mathbf{S} \otimes \mathbf{T})$ that endow $(\mathcal{S}\ell, \otimes, \mathbf{2})$ with a structure of monoidal category. Furthermore, it is a symmetric monoidal category, since there also exists a natural isomorphism $c : \mathbf{R} \otimes \mathbf{S} \to \mathbf{S} \otimes \mathbf{R}$, sending $x \otimes y$ to $y \otimes x$, for every $x \in R$ and $y \in S$, and satisfying the corresponding coherent axioms.

There are other remarkable properties of the monoidal structure of $\mathcal{S}\ell$ that are worth mentioning, although we will no make an extensive use of them. For instance, for a fixed an object $\mathbf{S} \in \mathcal{S}\ell$, the functor $\_ \otimes \mathbf{S} : \mathcal{S}\ell \to \mathcal{S}\ell$ is a left adjoint of the hom-functor $\mathcal{S}\ell(\mathbf{S}, \_) : \mathcal{S}\ell \to \mathcal{S}\ell$, since there exists a natural isomorphism:

$$\mathcal{S}\ell(\mathbf{R} \otimes \mathbf{S}, \mathbf{T}) \cong \mathcal{S}\ell\big(\mathbf{R}, \mathcal{S}\ell(\mathbf{S}, \mathbf{T})\big).$$

Another important property is that $\mathbf{2}$, the unit of the monoidal category, is a *dualizer* of $\mathcal{S}\ell$, since $\mathbf{R}^\partial \cong \mathcal{S}\ell(\mathbf{R}, \mathbf{2})$, and therefore, there exists an isomorphism $\mathbf{R} \otimes \mathbf{S} \cong \mathcal{S}\ell(\mathbf{R}, \mathbf{S}^\partial)^\partial$.

4 Quantaloids and modules over quantaloids

This section is devoted to the definition and study of the first properties of categories enriched over $\mathcal{S}\ell$ and enriched functors between them, which are known as *quantaloids* and *morphisms of quantaloids*, respectively, and the categories of modules over quantaloids. The book [Ros96] is a good compilation of many results about the Theory of Quantaloids, whereas the book [Kel05] is a well-known reference for enriched categories over arbitrary monoidal categories. In general, a category enriched over a monoidal category $\mathcal{V}$, or $\mathcal{V}$*-category*, is a category that has hom-sets in $\mathcal{V}$, and the composition operation (or *composition law*, as is called in [Kel05]) and the identity element, which are arrows in $\mathcal{V}$, render commutative certain diagrams called the *coherent axioms*. In the case of quantaloids, the definition can be simplified, and it takes the following form (see [Ros96]):

Definition 4.1. An *enriched category over* $\mathcal{S}\ell$, or *quantaloid*, is a locally small category $\mathcal{Q}$ such that,

(i) for every two objects $A, B \in \mathcal{Q}$, the set $\mathcal{Q}(A, B)$ of morphisms from A to B in $\mathcal{Q}$ is an object of $\mathcal{S}\ell$, that is, it is a complete lattice;
(ii) for every $A, B, C \in \mathcal{Q}$, the composition of morphisms in $\mathcal{Q}$, restricted to these hom-sets, is a biresiduated map $\mathcal{Q}(B, C) \times \mathcal{Q}(A, B) \to \mathcal{Q}(A, C)$.

It is then obvious that quantales correspond to quantaloids with just one object in the following sense: given a quantaloid $\mathcal{Q}$ with just one object $\star$, the set of endomorphisms $\mathcal{Q}(\star, \star)$ is a complete lattice, actually a quantale as the composition of endomorphisms is a biresiduated operation. On the other direction, given a quantale, a quantaloid can be defined with just one object and with set of morphisms the elements of the quantale. The composition of morphisms is defined as the product in the quantale.

In what follows we will use the notation $[A, B]$ for the hom-set $\mathcal{Q}(A, B)$, whenever the quantaloid $\mathcal{Q}$ could be understood from the context.

Remark 4.2. Note that the category $\mathcal{S}\ell$ is a quantaloid itself, since for every $\mathbf{R}, \mathbf{S} \in \mathcal{S}\ell$, the set of residuated maps $\mathcal{S}\ell(\mathbf{R}, \mathbf{S})$ is a complete lattice, and composition of residuated maps is a biresiduated operation. This justifies the use of the notation $[\mathbf{R}, \mathbf{S}]$ for $\mathcal{S}\ell(\mathbf{R}, \mathbf{S})$, whenever it is clear from the context.

As we mentioned, morphisms of quantaloids are the enriched functors between them. An analogous comment is suitable about the general definition of enriched functors between $\mathcal{V}$-categories and the definition that we give here for the special case of quantaloids, i.e., $\mathcal{S}\ell$-categories. The simpler form that the definition of the enriched functor between $\mathcal{S}\ell$-categories is the following.

Definition 4.3. If $\mathcal{Q}$ and $\mathcal{Q}'$ are two quantaloids, then a functor $T : \mathcal{Q} \to \mathcal{Q}'$ is *enriched*, or a *morphism of quantaloids*, if for every two objects $A, B \in \mathcal{Q}$, the restriction of T to $[A, B]$ is a residuated map $T{\restriction}_{[A,B]} : [A, B] \to [TA, TB]$.

As usual, a *natural map* between two morphisms of quantaloids $T, T' : \mathcal{Q} \to \mathcal{Q}'$ is a family $\alpha = \{\alpha_A \in [TA, T'A] : A \in \mathcal{Q}\}$ of morphisms in $\mathcal{Q}'$ such that for every $f : A \to B$ in $\mathcal{Q}$, the following diagram commutes:

$$\begin{tikzcd} TA \arrow[r, "\alpha_A"] \arrow[d, "T(f)"'] & T'A \arrow[d, "T'(f)"] \\ TB \arrow[r, "\alpha_B"] & T'B \end{tikzcd}$$

Note that this is a diagram in $\mathcal{Q}'$ and, in particular, if $\mathcal{Q}' = \mathcal{S}\ell$, then the objects are complete lattices, and the arrows are residuated maps.

In order to introduce modules over a quantaloid, we first recall the definition of a module over a quantale, as it is a particular case, when the quantaloid has exactly one object. This process of generalizing from quantales to quantaloids (i.e., categories enriched over $\mathcal{S}\ell$) and from modules over quantales to modules over quantaloids parallels the abstraction that goes from the notion of ring to the notion of preadditive category (i.e., a category enriched over the category of abelian groups $\mathcal{A}b$), and from the notion of module over a ring to the notion of module over a preadditive category $\mathcal{C}$ (i.e., an additive functor $T : \mathcal{C} \to \mathcal{A}b$).

Recall that a module over a quantale $\mathcal{A}$ is a pair $\langle \mathbf{R}, * \rangle$, where $\mathbf{R}$ is a complete lattice and $* : \mathbf{A} \times \mathbf{R} \to \mathbf{R}$ is a biresiduated map satisfying (M1) and (M2). Thus, modules over a quantale $\mathcal{A}$ are somehow "actions" of $\mathcal{A}$ on a complete lattice. It is well known that actions of a monoid M on a set X are in bijective correspondence with monoid homomorphisms of the form $M \to \mathcal{S}et(X, X)$, where $\mathcal{S}et(X, X)$ is the monoid of the endomaps of X; in that sense M-set s can be defined as monoid homomorphisms of the form $M \to \mathcal{S}et(X, X)$. Analogously, modules over a quantale $\mathcal{A}$ can be thought as morphisms of quantales of the form $\mathcal{A} \to \langle \mathcal{S}\ell(\mathbf{R}, \mathbf{R}), \circ, id_{\mathbf{R}} \rangle$ in the following way: first, the biresiduated product $* : \mathbf{A} \times \mathbf{R} \to \mathbf{R}$ corresponds to a residuated map $\mathbf{A} \to \mathcal{S}\ell(\mathbf{R}, \mathbf{R})$ via the bijections of Section 3:

$$* : \mathbf{A} \times \mathbf{R} \to \mathbf{R} \quad \sim \quad \mathbf{A} \otimes \mathbf{R} \to \mathbf{R} \quad \sim \quad \mathbf{A} \to \mathcal{S}\ell(\mathbf{R}, \mathbf{R}).$$

And, (M1) and (M2) correspond exactly to the property of $\langle A, \cdot, 1 \rangle \to \langle \mathcal{S}\ell(\mathbf{R}, \mathbf{R}), \circ, id_{\mathbf{R}} \rangle$ being a monoid homomorphism. In the same way as categories are an abstraction of monoids, and functors in turn an abstraction of homomorphisms of monoids, quantaloids are the categorical abstraction of quantales, and morphisms of quantaloids the abstraction of morphisms of quantales. This yields the following notion of module over a quantale.

Definition 4.4. Given a quantaloid $\mathcal{Q}$, a $\mathcal{Q}$-*module* is a morphism of quantaloids $T : \mathcal{Q} \to \mathcal{S}\ell$. The *universe* of a $\mathcal{Q}$-module T is the map $|T| : \mathrm{Obj}(\mathcal{Q}) \to \mathrm{Obj}(\mathcal{S}et)$, where for every $A \in \mathcal{Q}$, $|T|A = |TA|$ is the underling set of the complete lattice TA.

The display in Figure 1 of these notions helps us understand why the study of modules over quantaloids and in particular closure operators over them is simultaneously a generalization of the study of modules over quantales and their closure operators and the study of closure operators over $\mathcal{S}et$-valued functors, that is to say, π-institutions. In Section 2 we paid special attention to modules coming from M-set s, and in Section 12 we will describe in details modules induced by π-institutions.

The category $\mathcal{Q}$-Mod of $\mathcal{Q}$-modules is a full subcategory of the functor category $\mathcal{S}\ell^{\mathcal{Q}}$. Thus $\mathcal{Q}$-Mod has $\mathcal{Q}$-modules as objects and natural maps as morphisms. The morphisms in the

$$
\begin{tikzcd}
& \text{Quantaloid} & \\
\text{Category} \arrow[ur] & & \text{Quantale} \arrow[ul] \\
& \text{Monoid} \arrow[ul] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \begin{array}{c}\text{Module over}\\ \text{a quantaloid}\end{array} & \\
\begin{array}{c}\mathcal{S}et\text{-valued}\\ \text{functor}\end{array} \arrow[ur] & & \begin{array}{c}\text{Module over}\\ \text{a quantale}\end{array} \arrow[ul] \\
& M\text{-set} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Figure 1: The arrows from left to right represent an enrichment of the objects with a lattice structure and residuation properties, while the arrows from right to left represent the pass from one signature to multiple signatures.

category $\mathcal{Q}$-Mod are also called $\mathcal{Q}$-*morphisms*. Given two $\mathcal{Q}$-modules T and T', we denote the set of $\mathcal{Q}$-morphisms between them by $\mathrm{Hom}_{\mathcal{Q}}(T, T') = \mathcal{Q}\text{-}Mod(T, T')$. Let us fix the following notation: for every $\mathcal{Q}$-module T, every arrow $a : A \to B$ in $\mathcal{Q}$, and every element $x \in TA$,

$$a *_T x = T(a)x.$$

With this notation, the naturality property for a $\mathcal{Q}$-morphism $\alpha : T \to T'$ can be expressed in the following way: for every $a : A \to B$ in $\mathcal{Q}$ and every $x \in TA$,

$$\alpha_B(a *_T x) = a *_{T'} \alpha_A x.$$

Thus, for every $\mathcal{Q}$-module and every $A, B \in \mathcal{Q}$, we can consider $*_T$ as a map

$$*_T : [A, B] \times TA \to TB,$$

which is in particular biresiduated. In order to prove that, suppose that $a : A \to B$ is in $\mathcal{Q}$, $x \in TA$, and $y \in TB$. Let $f_{x,y} = \bigvee\{f \in [TA, TB] : fx \leqslant y\}$, where the supremum is taken in $[TA, TB]$. Hence, $f_{x,y} \in [TA, TB]$, and we have:

$$a *_T x \leqslant y \;\Leftrightarrow\; T(a)x \leqslant y \;\Leftrightarrow\; T(a) \leqslant f_{x,y} \;\Leftrightarrow\; a \leqslant T^+(f_{x,y}),$$

where $T^+ : [TA, TB] \to [A, B]$ is the residuum of $T : [A, B] \to [TA, TB]$. Furthermore,

$$a *_T x \leqslant y \;\Leftrightarrow\; T(a)x \leqslant y \;\Leftrightarrow\; x \leqslant T(a)^+ y.$$

Thus, $*_T$ has right and left residua, $/_T : TB \times TA \to [A, B]$ and $\backslash_T : [A, B] \times TB \to TA$, such that for every $a : A \to B$, $x \in TA$, and $y \in TB$,

$$a *_T x \leqslant y \;\Leftrightarrow\; a \leqslant y/_T x \;\Leftrightarrow\; x \leqslant a\backslash_T y.$$

Note that the left residuum, $a\backslash_T y = T(a)^+(y)$, only depends on the fact that T is an $\mathcal{S}\ell$-valued functor, whereas the right residuum $y/_T x$ depends moreover on the fact that T is enriched and, furthermore, on the existence of $f_{x,y}$, which is ensured because $[TA, TB]$ is a complete lattice.

"Substructures" of $\mathcal{Q}$-modules, that is $\mathcal{Q}$-submodules, are defined in a standard way:

Definition 4.5. If S and T are $\mathcal{Q}$-modules, then S is said to be a $\mathcal{Q}$-*submodule* of T, in symbols $S \leqslant T$, if and only if $|S| \leqslant |T|$, i.e., for every $A \in \mathcal{Q}$, $|SA| \subseteq |TA|$, and moreover the inclusions $e_A : |SA| \hookrightarrow |TA|$ are the components of a $\mathcal{Q}$-morphism $e : S \to T$.

Remark 4.6. Note that, if S and T are $\mathcal{Q}$-modules, then the conditions for S being a $\mathcal{Q}$-submodule of T can be rewritten in the following way: for every $A \in \mathcal{Q}$, we have $SA \leqslant TA$, that is, SA is a sub-join-complete lattice of TA, and for every $a : A \to B$ in $\mathcal{Q}$ and every $x \in SA$, $a *_S x = a *_T x$.

As we state in the following lemma that $\mathcal{Q}$-submodules are determined by their universe.

Lemma 4.7. *Let $\mathcal{Q}$ be a quantaloid, T a $\mathcal{Q}$-module, and $s : \mathrm{Obj}(\mathcal{Q}) \to \mathrm{Obj}(\mathcal{S}et)$ a map. Then, there exists a $\mathcal{Q}$-submodule S of T such that $|S| = s$ if and only if s satisfies:*

(i) for every $A \in \mathcal{Q}$, $sA \subseteq |TA|$ is closed under arbitrary joins (taken in TA),
(ii) for every $a : A \to B$ in $\mathcal{Q}$, and every $x \in sA$, $a *_T x \in sB$.

Moreover, if such an S exists then it is unique.

Remark 4.8. Note that, if $S, R \leqslant T$, then $S \leqslant R$ if and only if $|S| \leqslant |R|$.

The preceding lemma can be used, for instance, in order to prove that the following definition of the image of a $\mathcal{Q}$-morphism is correct, since it is easy to see that given a $\mathcal{Q}$-morphism $\alpha : T \to T'$, the map $s : \mathrm{Obj}(Q) \to \mathrm{Obj}(\mathcal{S}et)$ determined by $sA = \{\alpha_A x : x \in TA\}$ satisfies Conditions (i) and (ii) of the preceding lemma.

Definition 4.9. If $\alpha : T \to T'$ is a $\mathcal{Q}$-morphism, then we define the *image* of α as the $\mathcal{Q}$-submodule S of T' with universe determined by $|SA| = \{\alpha_A x : x \in TA\}$.

The following is an interesting result[2] that can be interpreted as saying that the process of constructing the category of modules over a quantaloid is somehow an internal operation in the category of quantaloids.

Proposition 4.10. *For every quantaloid $\mathcal{Q}$, the category $\mathcal{Q}$-Mod of $\mathcal{Q}$-modules is a quantaloid.*

Proof. If T, T' are two $\mathcal{Q}$-modules, we define the following order in $\mathrm{Hom}_{\mathcal{Q}}(T, T')$, the set of $\mathcal{Q}$-morphisms from T to T': for every $\alpha, \beta \in \mathrm{Hom}_{\mathcal{Q}}(T, T')$, we set $\alpha \leqslant \beta$ if for every $A \in \mathcal{Q}$, $\alpha_A \leqslant \beta_A$ in $[TA, T'A]$. Let us show that, with this order, the set of $\mathcal{Q}$-morphisms from T to T' is a complete lattice.

For every family $\{\alpha^i : T \to T'\}_{i \in I}$ of $\mathcal{Q}$-morphisms, let us define the transformation $\bigvee \alpha^i : T \to T'$ that has as components the following maps: for every $A \in \mathcal{Q}$, $\left(\bigvee \alpha^i\right)_A = \bigvee \alpha^i_A \in [TA, T'A]$. In order to prove the naturality of $\bigvee \alpha^i$, suppose that $x \in TA$, and $a : A \to B$ is in $\mathcal{Q}$. Therefore,

$$\left(\bigvee \alpha^i\right)_B (a *_T x) = \bigvee \alpha^i_B (a *_T x) = \bigvee \left(a *_{T'} \alpha^i_A x\right) = a *_{T'} \bigvee \alpha^i_A x = a *_{T'} \left(\bigvee \alpha^i\right)_A x.$$

We only need to prove now that the composition of $\mathcal{Q}$-morphisms

$$\mathrm{Hom}_{\mathcal{Q}}(T', T'') \times \mathrm{Hom}_{\mathcal{Q}}(T, T') \to \mathrm{Hom}_{\mathcal{Q}}(T, T'')$$

is biresiduated. This is a consequence of the fact that the operation of composition of residuated maps is biresiduated. Suppose that $\{\alpha^i : T' \to T''\}_{i \in I}$ is a family of $\mathcal{Q}$-morphisms and $\alpha : T \to T'$ is a $\mathcal{Q}$-morphism. Hence, for every $A \in \mathcal{Q}$,

$$\left(\left(\bigvee \alpha^i\right) \cdot \alpha\right)_A = \left(\bigvee \alpha^i\right)_A \circ \alpha_A = \left(\bigvee \alpha^i_A\right) \circ \alpha_A = \bigvee \left(\alpha^i_A \circ \alpha_A\right) = \bigvee (\alpha^i \cdot \alpha)_A = \left(\bigvee (\alpha^i \cdot \alpha)\right)_A$$

and hence $(\bigvee \alpha^i) \cdot \alpha = \bigvee (\alpha^i \cdot \alpha)$. Analogously for the other side. ☐

To end this section we define for every quantaloid $\mathcal{Q}$, a functor $Sub : \mathcal{Q}\text{-}Mod \to \mathcal{S}\ell$ in the following way. First, note that given a $\mathcal{Q}$-module T, the relation of "being a $\mathcal{Q}$-submodule" defines an order $\leqslant$ in the class $\mathrm{Sub}(T)$ of all the $\mathcal{Q}$-submodules of T, and that given a family $\mathcal{F} = \{S_i : i \in I\}$ of $\mathcal{Q}$-submodules of T, there exists the meet of $\mathcal{F}$ in $\langle \mathrm{Sub}(T), \leqslant \rangle$, and is determined by $|\bigwedge S_i| A = \bigcap |S_i A|$. Furthermore, given a $\mathcal{Q}$-morphism $\tau : T \to T'$ and a $\mathcal{Q}$-submodule S of T, the composition of the inclusion $e : S \hookrightarrow T$ with τ yields a $\mathcal{Q}$-morphism $\tau \cdot e : S \to T'$. We define the $\mathcal{Q}$-submodule $\overrightarrow{\tau}[S]$ of T' as the image of $\tau \cdot e$. It is straightforward to prove that, given two morphisms $\tau : T \to T'$ and $\rho : T' \to T''$, and a submodule S of T, then $\overrightarrow{\rho}\left[\overrightarrow{\tau}[S]\right] = \overrightarrow{\rho\tau}[S]$. And it is immediate that, if id_T is the identity on T, then

[2] This is Proposition 5.2.1 of [Ros96], that was stated without a proof.

$\overrightarrow{id_T}[S] = S$. Therefore, the correspondence expressed by the following diagram is a functor $Sub : \mathcal{Q}\text{-}Mod \to \mathcal{S}\ell$

$$\begin{tikzcd} T \arrow[r, mapsto] \arrow[d, "\tau"'] & \mathbf{Sub}(T) \arrow[d, "\overrightarrow{\tau}"] \\ T' \arrow[r, mapsto] & \mathbf{Sub}(T') \end{tikzcd}$$

Nevertheless, this functor is not in general a $(\mathcal{Q}\text{-}Mod)$-module, that is, $Sub : \mathcal{Q}\text{-}Mod \to \mathcal{S}\ell$ is not an enriched functor, in general.

5 Duality in the categories of $\mathcal{Q}$-modules

For every quantaloid $\mathcal{Q}$, the *opposite quantaloid* of $\mathcal{Q}$ is $\mathcal{Q}^{\mathrm{op}}$, the dual of $\mathcal{Q}$ as a category. That is to say, the objects of $\mathcal{Q}^{\mathrm{op}}$ are the same as the objects of $\mathcal{Q}$, and for every $A, B \in \mathcal{Q}^{\mathrm{op}}$, the set of morphisms from A to B in $\mathcal{Q}^{\mathrm{op}}$ is $\mathcal{Q}^{\mathrm{op}}(A, B) = \mathcal{Q}(B, A) = [B, A]$. Note that only the direction of the arrows is reversed, but not the lattice order. As usual, the composition of morphisms in $\mathcal{Q}^{\mathrm{op}}$ is defined reversing the order of the composition in $\mathcal{Q}$: if $a : A \to B$ and $b : B \to C$ are in $\mathcal{Q}^{\mathrm{op}}$, then $a : B \to A$ and $b : C \to B$ are in $\mathcal{Q}$, and hence $a \circ b : C \to A$ is in $\mathcal{Q}$, and therefore $a \circ b : A \to C$ is in $\mathcal{Q}^{\mathrm{op}}$. Thus, the composition in $\mathcal{Q}^{\mathrm{op}}$ of a and b is defined as $b \circ' a = a \circ b$.

The next proposition follows from a general result about enriched categories over symmetric monoidal categories (see [Kel05]), as indeed $\mathcal{S}\ell$ is.

Proposition 5.1. *For every quantaloid $\mathcal{Q}$, the opposite quantaloid $\mathcal{Q}^{\mathrm{op}}$ is also a quantaloid.*

Definition 5.2. Given a functor $T : \mathcal{Q} \to \mathcal{S}\ell$, its *dual*[3] is the functor $T^\partial : \mathcal{Q}^{\mathrm{op}} \to \mathcal{S}\ell$ determined by the following:

$$\begin{tikzcd} A \arrow[r, mapsto] \arrow[d, "a"'] & T^\partial(A) = (TA)^\partial \arrow[d, "T^\partial(a)=T(a)^+"] \\ B \arrow[r, mapsto] & T^\partial(B) = (TB)^\partial \end{tikzcd}$$

where $T(a)^+$ is the residuum of $T(a)$.

Note that, since $a : A \to B$ is an arrow in $\mathcal{Q}^{\mathrm{op}}$, then it is an arrow $a : B \to A$ in $\mathcal{Q}$, whence $T(a) : TB \to TA$ is a residuated map. Therefore, its residuum $T(a)^+$ is a residuated map between the duals: $T(a)^+ : (TA)^\partial \to (TB)^\partial$. Since the identity map is its own residuum, and the residuum of a composition of residuated maps is the composition of the residuated maps in the reversed order, it follows that $T^\partial : \mathcal{Q}^{\mathrm{op}} \to \mathcal{S}\ell$ is a functor.

Proposition 5.3. *The dual of a $\mathcal{Q}$-module is a $\mathcal{Q}^{\mathrm{op}}$-module.*

Proof. We have shown that for every functor $T : \mathcal{Q} \to \mathcal{S}\ell$, its dual $T^\partial : \mathcal{Q}^{\mathrm{op}} \to \mathcal{S}\ell$ is a functor. It only remains to prove that it is an enriched functor, whenever T is so. In order to prove that, suppose that $A, B \in \mathcal{Q}^{\mathrm{op}}$, and let us see that the restriction of T^∂ is a residuated map $\mathcal{Q}^{\mathrm{op}}(A, B) \to [T^\partial A, T^\partial B]$.

By hypothesis, the restriction of T to $[B, A] \to [TB, TA]$ is a residuated map and, as usual, let $T^+ : [TB, TA] \to [B, A]$ denote its residuum. We have that, for every $a \in \mathcal{Q}^{\mathrm{op}}(A, B)$ and every $f \in [T^\partial A, T^\partial B]$,

$$\begin{aligned} T^\partial(a) \leqslant f &\iff T(a)^+ \leqslant f && \text{in } [T^\partial A, T^\partial B] \\ &\iff T(a) \leqslant f^+ && \text{in } [TB, TA] \\ &\iff a \leqslant T^+(f^+) && \text{in } [B, A] = \mathcal{Q}^{\mathrm{op}}(A, B). \end{aligned}$$

[3]Note that this is not the dual functor $T^{\mathrm{op}} : \mathcal{Q}^{\mathrm{op}} \to \mathcal{S}\ell^{\mathrm{op}}$ of [Kel05].

Whence we obtain that the residuum of $T^\partial : \mathcal{Q}^{\mathrm{op}}(A,B) \to [T^\partial A, T^\partial B]$ is the map $[T^\partial A, T^\partial B] \to \mathcal{Q}^{\mathrm{op}}(A,B)$ determined by $f \mapsto T^+(f^+)$. $\square$

Remark 5.4. Note that, if T is a $\mathcal{Q}$-module, then for every $a : A \to B$ in $\mathcal{Q}^{\mathrm{op}}$ and $x \in T^\partial A$,

$$a *_{T^\partial} x = T^\partial(a)x = T(a)^+ x = a \backslash_T x.$$

The following is the main result of this section. It states the duality property for the categories of modules over quantaloids. Indeed, it says that the map $T \mapsto T^\partial$ extends to a dual isomorphism of categories from $\mathcal{Q}$-*Mod* to $\mathcal{Q}^{\mathrm{op}}$-*Mod*. The duality for modules over quantaloids is very useful, since if we prove that a categorical property is satisfied for all the categories of modules over quantaloids, then we will obtain that its dual is also satisfied for all the categories of modules over quantaloids.

Theorem 5.5. *For every quantaloid $\mathcal{Q}$, there exists an enriched dual isomorphism of categories*

$$(\_)^\partial : \mathcal{Q}\text{-}Mod \to \mathcal{Q}^{\mathrm{op}}\text{-}Mod.$$

Proof. We have already defined the functor $(\_)^\partial$ on objects. Let us define it on arrows. For every $\mathcal{Q}$-morphism $\alpha : T_1 \to T_2$, let α^∂ be the transformation with components the residua of the components of α. That is, for every $A \in \mathcal{Q}^{\mathrm{op}}$, $\alpha_A : T_1A \to T_2A$ is a residuated map with residuum $\alpha_A^+ : (T_2A)^\partial \to (T_1A)^\partial$. Thus, we take $(\alpha^\partial)_A = \alpha_A^+ : T_2^\partial A \to T_1^\partial A$. Let us see that $\alpha^\partial : T_2^\partial \to T_1^\partial$ is a natural transformation. Suppose that $a : A \to B$ is in $\mathcal{Q}^{\mathrm{op}}$. Therefore, $a : B \to A$ is in $\mathcal{Q}$, and by the naturality of α, the first of the following diagrams commutes, and this implies the commutativity of the second one:

$$\begin{tikzcd}
T_2A & T_1A \arrow[l, "\alpha_A"'] & & (T_2A)^\partial \arrow[r, "\alpha_A^+"] \arrow[d, "T_2(a)^+"'] & (T_1A)^\partial \arrow[d, "T_1(a)^+"] & & T_2^\partial A \arrow[r, "(\alpha^\partial)_A"] \arrow[d, "T_2^\partial(a)"'] & T_1^\partial A \arrow[d, "T_1^\partial(a)"] \\
T_2B \arrow[u, "T_2(a)"] & T_1B \arrow[l, "\alpha_B"'] \arrow[u, "T_1(a)"'] & \Rightarrow & (T_2B)^\partial \arrow[r, "\alpha_B^+"] & (T_1B)^\partial & = & T_2^\partial B \arrow[r, "(\alpha^\partial)_B"] & T_1^\partial B
\end{tikzcd}$$

It is easy to see that, if $\alpha : T \to T'$ and $\beta : T' \to T''$ are two $\mathcal{Q}$-morphisms, then $(\alpha\beta)^\partial = \beta^\partial\alpha^\partial$, since this equality is satisfied in every component. Therefore, $(\_)^\partial$ is a contravariant functor. It is straightforward to check that $\left((\_)^\partial\right)^\partial = Id$, which implies that $(\_)^\partial$ is a dual isomorphism of categories.

Finally, if $\{\alpha^i : T \to T'\}_{i \in I}$ is a family of $\mathcal{Q}$-morphisms and $A \in \mathcal{Q}$, we have the equalities:

$$\left(\left(\bigvee \alpha^i\right)^\partial\right)_A = \left(\left(\bigvee \alpha^i\right)_A\right)^+ = \left(\bigvee \alpha^i_A\right)^+ = \bigvee (\alpha^i_A)^+ = \bigvee \left((\alpha^i)^\partial\right)_A = \left(\bigvee (\alpha^i)^\partial\right)_A.$$

For the middle equality, recall that for every pair of complete lattices, $\mathbf{R}$ and $\mathbf{S}$, the map $(\_)^+ : [\mathbf{R}, \mathbf{S}] \to [\mathbf{S}^\partial, \mathbf{R}^\partial]$ is an isomorphism of complete lattices. Thus, $\left(\bigvee \alpha^i\right)^\partial = \bigvee (\alpha^i)^\partial$, which proves that the functor $(\_)^\partial$ is enriched. $\square$

Remark 5.6. Note that, in virtue of Propositions 5.1 and 4.10, another way to express the same is to say that, for every quantaloid $\mathcal{Q}$, "passing to the dual" is an isomorphism of quantaloids:

$$\mathcal{Q}^{\mathrm{op}}\text{-}Mod \cong (\mathcal{Q}\text{-}Mod)^{\mathrm{op}}.$$

Finally, to end this section, we obtain as a corollary of the duality property for the categories of modules over quantaloids a characterization for $\mathcal{Q}$-morphisms, which we will use later on.

Corollary 5.7. *If $\mathcal{Q}$ is a quantaloid, T and T' are two $\mathcal{Q}$-modules and $\tau = \{\tau_A : TA \to T'A\}_{A \in \mathcal{Q}}$ a family of residuated maps, then $\tau : T \to T'$ is a $\mathcal{Q}$-morphism if and only if for every $a : A \to B$ and every $y \in T'B$,*

$$\tau_A^+(a \backslash_{T'} y) = a \backslash_T \tau_A^+ y.$$

Proof. It is an immediate consequence of Theorem 5.5 and Remark 5.4. $\square$

6 Epis and monos in the categories of $\mathcal{Q}$-modules

The aim of the current section is to offer a characterization of epis and monos in the categories of modules over quantaloids. In the simpler case of modules over quantales, this can be done by studying first the $\mathcal{A}$-morphisms of the form $\mathbb{A} \to \mathbb{R}$, and finding that they are in a bijective correspondence with the elements of $\mathbb{R}$. In fact, this bijective correspondence is a toy version of the Yoneda Lemma, presented here in its version for modules over quantaloids.

Given a quantaloid $\mathcal{Q}$, there exists for every $A \in \mathcal{Q}$ a (covariant) hom-functor $h_A : \mathcal{Q} \to \mathcal{S}\ell$ that is determined by

$$h_A : \quad \begin{array}{ccc} B & \longmapsto & h_A B = [A, B] \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle h_A(a) = a \circ \_} \\ C & \longmapsto & h_A C = [A, C] \end{array}$$

where $h_A(a)$ is *postcomposition* with a, that is to say, for every $x \in [A, B]$, we have that $x : A \to B$, and $h_A(a)x = a \circ x$. This is trivially an enriched functor, since composition in $\mathcal{S}\ell$ is biresiduated. Thus, it is a $\mathcal{Q}$-module. We call h_A the $\mathcal{Q}$-module *associated* with A. Note that, with our notation, for every $a : B \to C$ in $\mathcal{Q}$ and every $x \in [A, B]$, we have $a *_{h_A} x = a \circ x$.

The following result is a version of the Yoneda Lemma strong enough to obtain the desired characterization of monos. For a more general version of the Yoneda Lemma we refer to [Kel05].

Theorem 6.1 (Yoneda Lemma)**.** *If $\mathcal{Q}$ is a quantaloid and T is a $\mathcal{Q}$-module, for every $A \in \mathcal{Q}$ there exists an isomorphism of complete lattices*

$$TA \cong \mathrm{Hom}_{\mathcal{Q}}(h_A, T),$$

which is natural in A and in T. □

That is to say, every $\mathcal{Q}$-morphism $h_A \to T$ is determined by an element of TA, and every element of TA determines a $\mathcal{Q}$-morphism $h_A \to T$, and these correspondences are monotone and bijective. We will not give a complete proof of the result, since it follows from a more general version (see [Kel05]). Nevertheless, let us see how these correspondences can be found.

Suppose that $\alpha : h_A \to T$ is a $\mathcal{Q}$-morphism. Hence, for every $B \in \mathcal{Q}$ and every $x \in h_A B$, we have that $x : A \to B$, and by the naturality of α:

$$\alpha_B x = \alpha_B(x \circ 1_A) = \alpha_B(x *_{h_A} 1_A) = x *_T \alpha_A(1_A),$$

and hence, α only depends on T and on the value of $\alpha_A(1_A) \in TA$.

Now, suppose that $t \in TA$ is fixed and define for every $B \in \mathcal{Q}$ and every $x \in h_A B$,

$$\mu^t_B x = x *_T t.$$

Let us see that $\mu^t : h_A \to T$ is a natural transformation. First note that, since T is a $\mathcal{Q}$-module, $\mu^t_B : h_A B \to TB$ is a residuated map. Suppose now that $a : B \to C$ is in $\mathcal{Q}$ and $x \in h_A B$. Then,

$$\mu^t_C(a *_{h_A} x) = (a *_{h_A} x) *_T t = (a \circ x) *_T t = a *_T (x *_T t) = a *_T \mu^t_B x,$$

whence we obtain the naturality of μ^t.

These two correspondences are bijective, since $\mu^t_A(1_A) = t$. Moreover, if $\{t_i : i \in I\}$ is a family of elements of TA, $B \in \mathcal{Q}$ and $x \in h_A B$, then

$$\mu^{\bigvee t_i}_B x = x *_T \bigvee t_i = \bigvee (x *_T t_i) = \bigvee \mu^{t_i}_B(x) = \big(\bigvee \mu^{t_i}\big)_B x,$$

whence we obtain $\mu^{\bigvee t_i} = \bigvee \mu^{t_i}$.

We use the Yoneda Lemma in order to characterize the monos in any category of $\mathcal{Q}$-modules, and as a corollary of this characterization and by the duality property of the categories of modules over quantaloids, we will also obtain a characterization of the epis.

Proposition 6.2. *If $\mathcal{Q}$ is a quantaloid, then the monos in the category $\mathcal{Q}$-Mod are those $\mathcal{Q}$-morphisms that are injective in every component.*

Proof. If $\alpha : T \to T'$ is a $\mathcal{Q}$-morphism with every component injective, then it is trivially a mono in the category of $\mathcal{Q}$-modules.

Suppose that $\alpha : T \to T'$ is a mono in $\mathcal{Q}$-Mod, let $A \in \mathcal{Q}$ arbitrary and $t_1, t_2 \in TA$ such that $\alpha_A(t_1) = \alpha_A(t_2) = t'$. Let us see that $t_1 = t_2$.

Let $\mu^{t_1}, \mu^{t_2} : h_A \to T$ be the $\mathcal{Q}$-morphisms given by the Yoneda Lemma. Then, the two compositions $\alpha \cdot \mu^{t_i} : h_A \to T'$, $i = 1, 2$, are $\mathcal{Q}$-morphisms, and by the Yoneda Lemma they are determined by their value at 1_A. Evaluating at 1_A we obtain

$$(\alpha \cdot \mu^{t_i})_A(1_A) = \alpha_A(\mu^{t_i}_A(1_A)) = \alpha_A(t_i) = t', \quad i = 1, 2.$$

Therefore, $\alpha \cdot \mu^{t_1} = \mu^{t'} = \alpha \cdot \mu^{t_2}$, and since α is supposed to be mono, $\mu^{t_1} = \mu^{t_2}$, whence we obtain $t_1 = t_2$. □

Corollary 6.3. *If $\mathcal{Q}$ is a quantaloid, then the epis in the category of $\mathcal{Q}$-Mod are exactly those $\mathcal{Q}$-morphisms that in every component are onto.*

Proof. Epis in the category $\mathcal{Q}$-Mod correspond by duality to monos in the category $\mathcal{Q}^{\mathrm{op}}$-$Mod$. Thus, $\beta : T_1 \twoheadrightarrow T_2$ is an epi in $\mathcal{Q}$-Mod if and only if $\beta^\partial : T_2^\partial \to T_1^\partial$ is a mono in $\mathcal{Q}^{\mathrm{op}}$-$Mod$, if and only if for every $A \in \mathcal{Q}^{\mathrm{op}}$, $\beta_A^+ : (T_2A)^\partial \to (T_1A)^\partial$ is injective, if and only if for every $A \in \mathcal{Q}$, $\beta_A : T_1A \to T_2A$ is onto, in virtue of a general result about residuated maps. □

7 Closure operators and closure systems in $\mathcal{Q}$-Mod

In this section we define in a natural way the notions of a closure operator and of a closure system on $\mathcal{Q}$-modules (where $\mathcal{Q}$ is a quantaloid). They generalize the notions of a closure operator and a closure system on modules over quantales in the sense that, if $\mathcal{Q}$ is a quantaloid with just one object, and T is a $\mathcal{Q}$-module, then a closure operator (closure system) on T as a module over a quantaloid corresponds to a closure operator (closure system) on T as a module over a quantale.

Definition 7.1. Let $\mathcal{Q}$ be a quantaloid and T a $\mathcal{Q}$-module. A *(structural) closure operator* on T is a family of closure operators $\gamma = \{\gamma_A : TA \to TA\}_{A \in \mathcal{Q}}$ such that for every $a : A \to B$ in $\mathcal{Q}$, and every $x \in TA$, the following property is satisfied:

$$a *_T \gamma_A(x) \leqslant \gamma_B(a *_T x). \tag{Str}$$

This property is called *structurality.* The set $\mathrm{Clop}(T)$ of closure operators on T is partially ordered by $\gamma \leqslant \gamma' \iff \forall A \in \mathcal{Q},\ \gamma_A \leqslant \gamma'_A$.

Remark 7.2. Note that, in general, closure operators are not $\mathcal{Q}$-morphisms; since the inequality of (Str) is not an equality, in general they are not natural maps. Moreover, the components of γ need not be residuated maps.

The following lemma exhibits some conditions that are equivalent to structurality. First of all, observe that (Str) can be rewritten in the following form: for every $a : A \to B$ in $\mathcal{Q}$, $T(a)\gamma_B \leqslant \gamma_A T(a)$. We will use the benefits of the functorial notation in the proof.

Lemma 7.3. *Let $\mathcal{Q}$ be a quantaloid, T a $\mathcal{Q}$-module and $\gamma = \{\gamma_A : TA \to TA\}_{A \in \mathcal{Q}}$ a family of closure operators. Then the following statements are equivalent, where $a : A \to B$ in $\mathcal{Q}$, $x \in TA$, and $y \in TB$ are arbitrary:*

(i) $a *_T \gamma_A x \leqslant \gamma_B(a *_T x)$,
(ii) $\gamma_B(a *_T \gamma_A x) = \gamma_B(a *_T x)$,
(iii) $a *_T \gamma_A(a \backslash_T y) \leqslant \gamma_B y$,
(iv) $\gamma_A(a \backslash_T y) \leqslant a \backslash_T \gamma_B y$,

(v) $\gamma_A(a\backslash_T\gamma_B y)=a\backslash_T\gamma_B y$.

Proof.

$$\begin{aligned}
(i)\Rightarrow(ii)\quad & \gamma_B T(a)\gamma_A\leqslant\gamma_B\gamma_B T(a)=\gamma_B T(a)\leqslant\gamma_B T(a)\gamma_A.\\
(ii)\Rightarrow(iii)\quad & T(a)\gamma_A T(a)^+\leqslant\gamma_B T(a)\gamma_A T(a)^+=\gamma_B T(a)T(a)^+\leqslant\gamma_B.\\
(iii)\Rightarrow(iv)\quad & \gamma_A T(a)^+\leqslant T(a)^+T(a)\gamma_A T(a)^+\leqslant T(a)^+\gamma_B.\\
(iv)\Rightarrow(v)\quad & \gamma_A T(a)^+\gamma_B\leqslant T(a)^+\gamma_B\gamma_B=T(a)^+\gamma_B\leqslant\gamma_A T(a)^+\gamma_B.\\
(v)\Rightarrow(i)\quad & T(a)\gamma_A\leqslant T(a)\gamma_A T(a)^+T(a)\leqslant T(a)\gamma_A T(a)^+\gamma_B T(a)=T(a)T(a)^+\gamma_B T(a)\\
& \leqslant\gamma_B T(a).
\end{aligned}$$

□

The following two lemmas are converse of each other. In the first we show that every $\mathcal{Q}$-morphism τ determines a closure operator $\widetilde{\tau}$, and in the second that every closure operator γ determines a $\mathcal{Q}$-morphism $\dot{\gamma}$. Furthermore, thanks to the characterization of Corollary 6.3, every morphism of the form $\dot{\gamma}$ is trivially an epi. This will be used later, in Proposition 7.7, where we prove that every epi β is isomorphic to $\dot{\widetilde{\beta}}$. This will be very useful in the next section to prove that every epi in $\mathcal{Q}$-Mod is regular, and applying duality, that every mono is regular.

Lemma 7.4. *Every $\mathcal{Q}$-morphism $\tau:T\to T'$ determines a closure operator $\widetilde{\tau}$ on T in the following way: $\widetilde{\tau}=\tau^{\partial}\tau$, that is, for every $A\in\mathcal{Q}$, $\widetilde{\tau}_A=\tau_A^+\tau_A$.*

Proof. Let $\tau:T\to T'$ be a $\mathcal{Q}$-morphism, and $\widetilde{\tau}$ defined as above. Since $\tau_A:TA\to T'A$ is a residuated map, it is obvious that for every $A\in\mathcal{Q}$, $\widetilde{\tau}_A$ is a closure operator on TA with associated closure system $\{\tau_A^+y:y\in T'A\}$ (this follows from a general result about residuated maps). In order to prove the structurality property for $\widetilde{\tau}$, suppose that $a:A\to B$ is in $\mathcal{Q}$, and $x\in TA$. Then, we have the following implications:

$$\begin{aligned}
& a*_{T'}\tau_A\tau_A^+\tau_A x=a*_{T'}\tau_A x\ \Leftrightarrow\\
&\qquad\Leftrightarrow\ \tau_B\big(a*_T\tau_A^+\tau_A(x)\big)=\tau_B(a*_T x), && \text{by the naturality of }\tau,\\
&\qquad\Rightarrow\ a*_T\tau_A^+\tau_A(x)\leqslant\tau_B^+\tau_B(a*_T x), && \text{since }\tau_B\text{ is residuated},\\
&\qquad\Leftrightarrow\ a*_T\widetilde{\tau}_A(x)\leqslant\widetilde{\tau}_B(a*_T x), && \text{by the definition of }\widetilde{\tau}.
\end{aligned}$$

Since τ_A is a residuated map, the first equality is true, and so is the last, which is the structurality property for τ, since $a:A\to B$ and $x\in TA$ are arbitrary. □

Definition 7.5. Let T be a $\mathcal{Q}$-module and γ a closure operator on T. We define the functor $T_\gamma:\mathcal{Q}\to\mathcal{S}\ell$ as follows:

(i) for every $A\in\mathcal{Q}$, $T_\gamma A=(TA)_{\gamma_A}$ is the complete lattice $\mathbf{Cl}(\gamma_A)$ of γ_A-closed sets, that is, $T_\gamma A=\{\gamma_A x:x\in A\}=\{x\in TA:x=\gamma_A x\}$, with the induced order;
(ii) for every $a:A\to B$ in $\mathcal{Q}$, and every $x\in TA$, $T_\gamma(a)x=\gamma_B(a*_T x)$.

This is a $\mathcal{Q}$-module, as we prove in the next lemma, and we call it the $\mathcal{Q}$-module *associated* with the closure operator γ.

Lemma 7.6. *Let $\mathcal{Q}$ be a quantaloid and T a $\mathcal{Q}$-module. Then every closure operator γ on T determines an epi $\mathcal{Q}$-morphism $\dot{\gamma}:T\twoheadrightarrow T_\gamma$.*

Proof. Let us first prove that T_γ is a $\mathcal{Q}$-module: note that, for every $A\in\mathcal{Q}$, $T_\gamma(1_A)=id_{T_\gamma A}$, since γ_A is idempotent. Suppose now that $a:A\to B$ and $b:B\to C$ are in $\mathcal{Q}$. Then, using Lemma 7.3, for every $x\in T_\gamma A$,

$$\begin{aligned}
T_\gamma(ba)x&=\gamma_C\big((ba)*_T x\big)=\gamma_C\big(b*_T(a*_T x)\big)=\gamma_C\big(b*_T\gamma_B(a*_T x)\big)\\
&=T_\gamma(b)\big(T_\gamma(a)x\big)=\big(T_\gamma(b)T_\gamma(a)\big)x.
\end{aligned}$$

Let us see that for every $a:A\to B$ in $\mathcal{Q}$, the map $T_\gamma(a):T_\gamma A\to T_\gamma B$ is residuated. If we suppose that $x\in T_\gamma A$ and $y\in T_\gamma B$, then

$$T_\gamma(a)x\leqslant y\ \Leftrightarrow\ \gamma_B(a*_T x)\leqslant y\ \Leftrightarrow\ a*_T x\leqslant y\ \Leftrightarrow\ x\leqslant a\backslash_T y.$$

And finally, let us prove that the restriction of T_γ to $[A,B] \to [T_\gamma A, T_\gamma B]$ is residuated. We know that the restriction of T to $[A,B] \to [TA, TB]$ is residuated. Let $T^+ : [TA, TB] \to [A,B]$ denote its residuum. Then, if $a \in [A,B]$ and $f \in [T_\gamma A, T_\gamma B]$,

$$\begin{aligned} T_\gamma(a) \leqslant f \;\Leftrightarrow\; \forall x \in T_\gamma A,\; \gamma_A(a *_T x) \leqslant fx \;\Leftrightarrow\; \forall x \in T_\gamma A,\; a *_T x \leqslant fx \\ \Leftrightarrow\; T(a) \leqslant f \;\Leftrightarrow\; a \leqslant T^+(f). \end{aligned}$$

Therefore, T_γ is a $\mathcal{Q}$-module. If for every $A \in \mathcal{Q}$, $\dot\gamma_A$ is the restriction of γ_A to its image, $\dot\gamma_A : TA \to (TA)_{\gamma_A}$, then it is known that it is a residuated map. Let us see that, moreover, $\dot\gamma : T \to T_\gamma$ is an epi $\mathcal{Q}$-morphism. Since every $\dot\gamma_A$ is onto, the only thing we have to check is that, for every $a : A \to B$ in $\mathcal{Q}$ and every $x \in TA$, $a *_{T_\gamma} \dot\gamma_A(x) = \dot\gamma_B(a *_T x)$. Note that, since $\dot\gamma_A$ is the restriction of γ_A to its image and γ is a closure operator on T, by Lemma 7.3,

$$a *_{T_\gamma} \dot\gamma_A x = \gamma_B(a *_T \gamma_A x) = \gamma_B(a *_T x) = \dot\gamma_B(a *_T x). \qquad \square$$

Thus, for every closure operator γ on a $\mathcal{Q}$-module T, we have constructed a $\mathcal{Q}$-module T_γ and an epimorphism $\dot\gamma : T \twoheadrightarrow T_\gamma$. The next proposition says that, indeed, every epimorphism in $\mathcal{Q}$-Mod is in essence the $\mathcal{Q}$-morphism associated with a closure operator.

Proposition 7.7. *Let $\mathcal{Q}$ be a quantaloid, T a $\mathcal{Q}$-module and $\beta : T \twoheadrightarrow T'$ an epi in $\mathcal{Q}$-Mod, $\gamma = \widetilde{\beta}$ the closure operator on T associated with β. Then β and $\dot\gamma$ are isomorphic arrows in $\mathcal{Q}$-Mod, that is, there exists an isomorphism $T_\gamma \to T'$ such that the following diagram commutes:*

$$\begin{tikzcd} & T \arrow[dl, "\dot\gamma"'] \arrow[dr, "\beta"] & \\ T_\gamma \arrow[rr, dashed, "\cong"] & & T' \end{tikzcd}$$

Proof. Let $\eta : T_\gamma \to T'$ be defined in every component $\eta_A : T_\gamma A \to T'A$ as the restriction of β_A to $(TA)_{\gamma_A}$. Recall that, for every $A \in \mathcal{Q}$, the closure system associated with the closure operator γ_A is the set $(TA)_{\gamma_A} = \{\beta_A^+ y : y \in T'A\}$. Note that, for every $x \in TA$, $\beta_A x = \beta_A \beta_A^+ \beta_A x = \beta_A \gamma_A x = \eta_A \gamma_A x$. Hence the following diagram commutes:

$$\begin{tikzcd} & TA \arrow[dl, "\dot\gamma_A"'] \arrow[dr, "\beta_A"] & \\ T_\gamma A \arrow[rr, dashed, "\eta_A"] & & T'A \end{tikzcd}$$

The map η_A is injective, since if $x, y \in T_\gamma A$ are such that $\eta_A x = \eta_A(x')$, then $\beta_A x = \beta_A(x')$, and hence $x = \gamma_A x = \beta_A^+ \beta_A x = \beta_A^+ \beta_A(x') = \gamma_A(x') = x'$. The map η_A is onto because for every $y \in T'A$, $\eta_A(\beta_A^+ y) = \beta_A \beta_A^+ y = y$, since β_A is onto.

The map $\eta_A : T_\gamma A \to T'A$ is residuated, since for every $x \in T_\gamma A$ and every $y \in T'A$, $\eta_A x \leqslant y \;\Leftrightarrow\; \beta_A x \leqslant y \;\Leftrightarrow\; x \leqslant \beta_A^+ y$, and then the residuum $\eta_A^+ : T'A \to T_\gamma A$ of η_A is the restriction of β_A^+ to $T_\gamma A$ in the codomain.

Finally, let us prove that $\eta : T_\gamma \to T'$ is natural. Suppose that $a : A \to B$ is in $\mathcal{Q}$, and $x \in T_\gamma A$. Taking into account for the middle equation that $\beta_B = \beta_B \beta_B^+ \beta_B = \beta_B \gamma_B$, we have:

$$a *_{T'} \eta_A x = a *_{T'} \beta_A x = \beta_B(a *_T x) = \beta_B \gamma_B(a *_T x) = \beta_B(a *_{T_\gamma} x) = \eta_B(a *_{T_\gamma} x). \qquad \square$$

Now we are looking for a definition of a closure system on a $\mathcal{Q}$-module in such a way that it gives us a bijective correspondence between closure systems and closure operators. We will obtain a result relating the closure systems on a $\mathcal{Q}$-module with the $\mathcal{Q}^{\mathrm{op}}$-submodules of its dual.

Definition 7.8. If T is a $\mathcal{Q}$-module, a *closure system* on T is a map $K : \mathrm{Obj}(\mathcal{Q}) \to \mathrm{Obj}(\mathcal{S}et)$ such that for every $A \in \mathcal{Q}$, $KA \subseteq |TA|$ is closed under arbitrary meets, and moreover, for every $a : A \to B$ in $\mathcal{Q}$ and every $y \in KB$, $a \backslash_T y \in KA$.

The set $\mathrm{Clsy}(T)$ of closure systems on a $\mathcal{Q}$-module T is evidently partially ordered by the relation $K \leqslant S \;\Leftrightarrow\; \forall A \in \mathcal{Q},\; KA \subseteq SA$, and $\mathbf{Clsy}(T) = \langle \mathrm{Clsy}(T), \leqslant \rangle$ is a complete lattice, since for every family $\{K_i : i \in I\}$ of closure systems on T, the meet of this family is the map $\bigwedge K_i$ determined by $(\bigwedge K_i)A = \bigcap K_i A$. It is very easy to see that this is a closure system.

It is easy to see using Lemma 7.3 that for every $\mathcal{Q}$-module T and every closure operator γ on T, the universe of T_γ is a closure system $|T_\gamma|$ on T. On the other direction, if K is a closure system on T, then for every $A \in \mathcal{Q}$, the set KA is a closure system on TA. Taking γ_K as the family of all the closure operators associated with these closure systems, $\gamma_K = \{\gamma_{KA} : A \in \mathcal{Q}\}$, we obtain a closure operator on T. The structurality property for γ_K is evident again in virtue of Lemma 7.3, since if $a : A \to B$ is in $\mathcal{Q}$ and $y \in TB$, then $\gamma_{KB}(y) \in KB$, and therefore $a \backslash_T \gamma_{KB}(y) \in KA$, whence we obtain $\gamma_{KA}\big(a \backslash_T \gamma_{KB}(y)\big) = a \backslash_T \gamma_{KB}(y)$.

These correspondences are inverse to each other, and furthermore for every $\gamma, \gamma' \in \mathrm{Clop}(T)$, $\gamma \leqslant \gamma' \;\Leftrightarrow\; |T_{\gamma'}| \leqslant |T_\gamma|$. Therefore, as it was expected, there is an isomorphism of lattices $|T_{\_}| : \mathbf{Clop}(T) \cong \mathbf{Clsy}(T)^\partial$, whose inverse is $\gamma_{\_} : \mathbf{Clsy}(T)^\partial \cong \mathbf{Clop}(T)$.

We can readily prove the following proposition. It relates closure systems on a $\mathcal{Q}$-module and $\mathcal{Q}^{\mathrm{op}}$-submodules of its dual. We will use these correspondences between closure operators, closure systems, and submodules, and the functor *Sub* of submodules, in order to define the functor *Clop* of closure operators and the functor *Clsy* of closure systems and prove some natural isomorphisms between them.

Proposition 7.9. *Let T be a $\mathcal{Q}$-module, and $K : \mathrm{Obj}(\mathcal{Q}) \to \mathrm{Obj}(\mathcal{S}et)$ a map. Then, K is a closure system on T if and only if K is the universe of a $\mathcal{Q}^\partial$-submodule $\langle K \rangle$ of T^∂. Moreover, the correspondences $|\_|_T : \mathbf{Sub}(T^\partial) \to \mathbf{Clsy}(T)$ and $\langle \_ \rangle_T : \mathbf{Clsy}(T) \to \mathbf{Sub}(T^\partial)$ are isomorphisms of complete lattices inverse to each other.*

Proof. This is a consequence of Lemma 4.7 and Remark 5.4. □

Remark 7.10. We wrote the subindexes of $|\_|_T$ and $\langle \_ \rangle_T$ in the previous proposition just to emphasize that these maps depend on the $\mathcal{Q}$-module T.

Thus, we have the isomorphisms:

$$\mathbf{Clop}(T) \cong \mathbf{Clsy}(T)^\partial \quad \text{and} \quad \mathbf{Clsy}(T) \cong \mathbf{Sub}(T^\partial). \tag{3}$$

Given a $\mathcal{Q}$-morphism τ between two $\mathcal{Q}$-modules, the direct image map $\overrightarrow{\tau}$ is a residuated map between their lattices of submodules with residuum the inverse image map $\overleftarrow{\tau}$. In what follows we are going to use the ismomorphisms (3) in order to define residuated maps between the lattices of closure operators and closure systems induced by a $\mathcal{Q}$-morphism $\tau : T \to T'$:

$$\begin{aligned}
&(_)^\tau : \mathbf{Clsy}(T') \to \mathbf{Clsy}(T), && \text{with residuum } {}^\tau(_) : \mathbf{Clsy}(T) \to \mathbf{Clsy}(T'),\\
&{}^\tau(_) : \mathbf{Clop}(T) \to \mathbf{Clop}(T'), && \text{with residuum } (_)^\tau : \mathbf{Clop}(T') \to \mathbf{Clop}(T).
\end{aligned}$$

Later on, we will give a detailed description of these maps (see Propositions 7.14 and 7.15.)

In the case of the closure systems, it is as follows: Let $\mathcal{Q}$ be a quantaloid and $\tau : T_1 \to T_2$ a $\mathcal{Q}$-morphism. The dual of τ is a $\mathcal{Q}^{\mathrm{op}}$-morphism $\tau^\partial : T_2^\partial \to T_1^\partial$, and then $\overrightarrow{\tau^\partial} : \mathbf{Sub}(T_2^\partial) \to \mathbf{Sub}(T_1^\partial)$ is a residuated map. Composing with the isomorphisms of the preceding proposition, we obtain a residuated map

$$(\_)^\tau : \; \mathbf{Clsy}(T_2) \xrightarrow{\langle \_ \rangle_{T_2}} \mathbf{Sub}(T_2^\partial) \xrightarrow{\overrightarrow{\tau^\partial}} \mathbf{Sub}(T_1^\partial) \xrightarrow{|\_|_{T_1}} \mathbf{Clsy}(T_1)$$

And, its residuum is given by:

$${}^\tau(\_) : \; \mathbf{Clsy}(T_1) \xrightarrow{\langle \_ \rangle_{T_1}} \mathbf{Sub}(T_1^\partial) \xrightarrow{\overleftarrow{\tau^\partial}} \mathbf{Sub}(T_2^\partial) \xrightarrow{|\_|_{T_2}} \mathbf{Clsy}(T_2)$$

This map ${}^{\tau}(\_)$ is a residuated map when considered between the duals of $\mathbf{Clsy}(T_1)$ and $\mathbf{Clsy}(T_2)$, and using the isomorphisms (3) we can define the residuated map between the lattices of closure operators as follows:

$$\mathbf{Clop}(T_1) \xrightarrow{K_{T_1}} \mathbf{Clsy}(T_1)^{\partial} \xrightarrow{{}^{\tau}(\_)} \mathbf{Clsy}(T_2)^{\partial} \xrightarrow{K_{T_2}^{-1}} \mathbf{Clop}(T_2)$$

where, for every closure operator γ on T, $K_T(\gamma) = |T_\gamma|$, the closure system associted with γ. The residuum of this map is:

$$\mathbf{Clop}(T_2) \xrightarrow{K_{T_2}} \mathbf{Clsy}(T_2)^{\partial} \xrightarrow{(\_)^{\tau}} \mathbf{Clsy}(T_1)^{\partial} \xrightarrow{K_{T_1}^{-1}} \mathbf{Clop}(T_1)$$

Thus, it is evident form the definitions that for every $\mathcal{Q}$-morphism $\tau : T_1 \to T_2$ the following diagrams commute:

$$\begin{tikzcd}
\mathbf{Clsy}(T_2) \arrow[r, "\langle_\rangle_{T_2}"] \arrow[d, "(_)^{\tau}"'] & \mathbf{Sub}(T_2^{\partial}) \arrow[d, "\overrightarrow{\tau^{\partial}}"] \\
\mathbf{Clsy}(T_1) \arrow[r, "\langle_\rangle_{T_1}"] & \mathbf{Sub}(T_1^{\partial})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Clop}(T_1) \arrow[r, "K_{T_1}"] \arrow[d, "{}^{\tau}(_)"'] & \mathbf{Clsy}(T_1)^{\partial} \arrow[d, "{}^{\tau}(_)"] \\
\mathbf{Clop}(T_2) \arrow[r, "K_{T_2}"] & \mathbf{Clsy}(T_2)^{\partial}
\end{tikzcd}
\tag{4}$$

This suggests that the isomorphisms (3) are natural in some sense. To make this precise we define the functors *Clsy* and *Clop* as follows: $Clsy : (\mathcal{Q}\text{-}Mod)^{\mathrm{op}} \to \mathcal{S}\ell$ maps every arrow τ from T_2 to T_1 in $(\mathcal{Q}\text{-}Mod)^{\mathrm{op}}$, that is, every $\mathcal{Q}$-morphism $\tau : T_1 \to T_2$, into $Clsy(\tau) = (\_)^{\tau} : \mathbf{Clsy}(T_2) \to \mathbf{Clsy}(T_1)$. And the functor $Clop : \mathcal{Q}\text{-}Mod \to \mathcal{S}\ell$ maps $\tau : T_1 \to T_2$ into $Clop(\tau) = {}^{\tau}(\_) : \mathbf{Clop}(T_1) \to \mathbf{Clop}(T_2)$

$$\begin{tikzcd}
T_1 \arrow[r, mapsto] \arrow[d, "\tau"'] & \mathbf{Clsy}(T_1) \\
T_2 \arrow[r, mapsto] & \mathbf{Clsy}(T_2) \arrow[u, "(_)^{\tau}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
T_1 \arrow[r, mapsto] \arrow[d, "\tau"'] & \mathbf{Clop}(T_1) \arrow[d, "{}^{\tau}(_)"] \\
T_2 \arrow[r, mapsto] & \mathbf{Clop}(T_2)
\end{tikzcd}$$

In order to show that *Clsy* is a functor, suppose that $\tau : T_1 \to T_2$ and $\rho : T_2 \to T_3$ are in $(\mathcal{Q}\text{-}Mod)^{\mathrm{op}}$, that is $\tau : T_2 \to T_1$ and $\rho : T_3 \to T_2$ are $\mathcal{Q}$-morphisms. Recall that the composition of morphisms in $(\mathcal{Q}\text{-}Mod)^{\mathrm{op}}$ is the reversed of the composition in $\mathcal{Q}\text{-}Mod$. Let us denote it by $\cdot'$ for a better understanding of the final of the following equalities, which show that *Clsy* respects composition.

$$\begin{aligned}
Clsy(\rho)Clsy(\tau) = (_)^{\rho}(_)^{\tau} &= \big(|_|_{T_3}\overrightarrow{\rho^{\partial}}\langle_\rangle_{T_2}\big)\big(|_|_{T_2}\overrightarrow{\tau^{\partial}}\langle_\rangle_{T_1}\big) = |_|_{T_3}\overrightarrow{\rho^{\partial}}\overrightarrow{\tau^{\partial}}\langle_\rangle_{T_1} \\
&= |_|_{T_3}\overrightarrow{\rho^{\partial}\tau^{\partial}}\langle_\rangle_{T_1} = |_|_{T_3}\overrightarrow{(\tau\rho)^{\partial}}\langle_\rangle_{T_1} = Clsy(\rho \cdot' \tau).
\end{aligned}$$

It is immediate that *Clsy* also respects the identities, and thus is a functor. Observe that the dual of the functor $Clsy : (\mathcal{Q}\text{-}Mod)^{\mathrm{op}} \to \mathcal{S}\ell$ is $Clsy^{\partial} : \mathcal{Q}\text{-}Mod \to \mathcal{S}\ell$, which applies every $\mathcal{Q}$-morphism $\tau : T_1 \to T_2$ into ${}^{\tau}(\_) : \mathbf{Clsy}(T_1)^{\partial} \to \mathbf{Clsy}(T_2)^{\partial}$.

Hence, the commutativity of the diagrams (4) ensures that $\langle\_\rangle : Clsy \dot{\to} Sub \circ (\_)^{\partial}$ and $K_{\_} : Clop \dot{\to} Clsy^{\partial}$ are natural isomorphism, where $(\_)^{\partial} : (\mathcal{Q}\text{-}Mod)^{\mathrm{op}} \to \mathcal{Q}^{\mathrm{op}}\text{-}Mod$ is the duality functor.

$$\begin{tikzcd}
(\mathcal{Q}\text{-}Mod)^{\mathrm{op}} \arrow[rr, bend left, "Clsy"{name=U}] \arrow[rr, bend right=15, ""{name=D}] \arrow[dr, "(_)^{\partial}"'] & & \mathcal{S}\ell \\
& \mathcal{Q}^{\mathrm{op}}\text{-}Mod \arrow[ur, "Sub"'] &
\arrow[Rightarrow, from=U, to=D, "\cong\langle_\rangle"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{Q}\text{-}Mod \arrow[r, bend left, "Clop"{name=U}] \arrow[r, bend right, "Clsy^{\partial}"'{name=D}] & \mathcal{S}\ell
\arrow[Rightarrow, from=U, to=D, "\cong K_{_}"]
\end{tikzcd}$$

We summarize all this in the following proposition.

Proposition 7.11. *Let $\mathcal{Q}$ be a quantaloid, and Sub : $\mathcal{Q}^{\rm op}$-Mod $\to \mathcal{S}\ell$, Clsy : $(\mathcal{Q}\text{-}Mod)^{\rm op} \to \mathcal{S}\ell$, and Clop : $\mathcal{Q}$-Mod $\to \mathcal{S}\ell$ the functors of submodules, closure systems, and closure operators, respectively. Then there exist natural isomorphisms:*

(i) $\langle\_\rangle : Clsy \overset{\cdot}{\to} Sub \circ (\_)^\partial$, *where* $(\_)^\partial : (\mathcal{Q}\text{-}Mod)^{\rm op} \to \mathcal{Q}^{\rm op}\text{-}Mod$ *is the duality isomorphism;*

(ii) $K_{\_} : Clop \overset{\cdot}{\to} Clsy^\partial$*;*

(iii) $Clop^\partial \overset{\cdot}{\to} Sub \circ (\_)^\partial$.

Proof. We have already proved the existence of the two first natural isomorphisms. For the third, note that since $K_{\_} : Clop \overset{\cdot}{\to} Clsy^\partial$ is a natural isomorphism, its dual, which coincides with its inverse, is a natural isomorphism $(K_{\_})^{-1} : Clsy \overset{\cdot}{\to} Clop^\partial$. And therefore, $K_{\_} : Clop^\partial \to Clsy$ is also a natural isomorphism. Composing with $\langle\_\rangle$, we obtain the natural isomorphism between $Clop^\partial$ and $Sub \circ (\_)^\partial$. □

Corollary 7.12. *Let $\mathcal{Q}$ be a quantaloid and T a $\mathcal{Q}$-module. Then for every closure operator γ on T, T_γ^∂ is a $\mathcal{Q}^{\rm op}$-submodule of T^∂. Moreover, the dual of the map $\dot\gamma : T \twoheadrightarrow T_\gamma$ is the inclusion $i : T_\gamma^\partial \hookrightarrow T^\partial$.*

Proof. If γ is a closure operator on T, then $\langle K_T(\gamma)\rangle$ is the unique submodule of T^∂ with universe $K_T(\gamma) = |T_\gamma|$. It is evident that this is T_γ^∂. □

Remark 7.13. Note that, in view of this corollary, Proposition 7.7 has a new interpretation: in the category $\mathcal{Q}$-Mod, the monos are essentially inclusions. That is, for every mono $\eta : T \to T'$, there exists a $\mathcal{Q}^{\rm op}$-submodule $S \leqslant T'$ and an isomorphism $\nu : T \to S$ such that $i\nu = \eta$. It can be proved that S is exactly the image of η, and that ν is the restriction of η in the target.

In what follows we give an easier description of the action of the functor *Clop* on the $\mathcal{Q}$-morphisms by describing for every $\mathcal{Q}$-morphism $\tau : T \to T'$ and closure operators $\gamma \in \mathrm{Clop}(T)$ and $\delta \in \mathrm{Clop}(T')$, the closure systems associated with ${}^\tau\gamma$ and δ^τ.

Proposition 7.14. *Let $\mathcal{Q}$ be a quantaloid, $\tau : T \to T'$ a $\mathcal{Q}$-morphism, $\gamma \in \mathrm{Clop}(T)$ and $\delta \in \mathrm{Clop}(T')$. Then ${}^\tau\gamma$ and δ^τ are the closure operators with associated closure systems determined in every component by $\{y \in T'A : \tau_A^+ y \in T_\gamma A\}$ and $\{\tau_A^+ y : y \in T'_\delta A\}$, respectively.*

Proof. The closure operator ${}^\tau\gamma$ has as associated closure system

$$K_{T'}({}^\tau\gamma) = {}^\tau\big(K_T(\gamma)\big) = \Big|\overleftarrow{\tau^\partial}\langle K_T(\gamma)\rangle\Big| = \Big|\overleftarrow{\tau^\partial}\big[T_\gamma^\partial\big]\Big|.$$

Then, for every $A \in \mathcal{Q}$, the closure system associated with the closure operator ${}^\tau\gamma_A$ is

$$\Big|\overleftarrow{\tau^\partial}\big[T_\gamma^\partial\big]\Big|A = \overleftarrow{\tau_A^+}\big((TA)_{\gamma_A}\big) = \{y \in T'A : \tau_A^+ y \in T_\gamma A\}.$$

Analogously, the closure operator δ^τ has as associated closure system

$$K_T(\delta^\tau) = \big(K_{T'}(\delta)\big)^\tau = \Big|\overrightarrow{\tau^\partial}\langle K_{T'}(\delta)\rangle\Big| = \Big|\overrightarrow{\tau^\partial}\big[(T'_\delta)^\partial\big]\Big|.$$

Thus, for every $A \in \mathcal{Q}$, the closure system associated with the closure operator δ^τ is

$$\Big|\overrightarrow{\tau^\partial}\big[(T'_\delta)^\partial\big]\Big|A = \overrightarrow{\tau_A^+}\big((T'A)_{\delta_A}\big) = \{\tau_A^+ y : y \in T'_\delta A\}.$$ □

If $\tau : T \to T'$ is a $\mathcal{Q}$-morphism and δ is a closure operator on T', we call the closure operator δ^τ the τ-*transform* of δ. We end this section with a more familiar description of this operator.

Proposition 7.15. *Let $\mathcal{Q}$ be a quantaloid, $\tau : T \to T'$ a $\mathcal{Q}$-morphism between two $\mathcal{Q}$-modules, and δ a closure operator on T'. Then the τ-transform of δ is $\delta^\tau = \tau^\partial\delta\tau$.*

Proof. Let $A \in \mathcal{Q}$, and $x \in TA$. Hence,

$$\begin{aligned}\delta_A^\tau x &= \bigwedge\Big\{t \in \overrightarrow{\tau_A^+}(T'_\delta A) : x \leqslant t\Big\} = \bigwedge\{\tau_A^+ y : y \in T'_\delta A,\ x \leqslant \tau_A^+ y\} \\ &= \bigwedge\{\tau_A^+ y : y \in T'_\delta A,\ \tau_A x \leqslant y\} = \tau_A^+\big(\bigwedge\{y \in T'_\delta A : \tau_A x \leqslant y\}\big) \\ &= \tau_A^+\big(\delta_A(\tau_A x)\big).\end{aligned}$$ □

8 Strong completeness and cocompleteness and factorizations of $\mathcal{Q}$-Mod

In this section we first prove that for every quantaloid $\mathcal{Q}$, the category $\mathcal{Q}$-Mod has biproducts and equalizers. This ensures the completeness of $\mathcal{Q}$-Mod, and its cocompleteness by duality. Therefore, $\mathcal{Q}$-Mod has pullbacks, and in particular congruences of $\mathcal{Q}$-morphisms. After that, we prove a fundamental result, Proposition 8.3, stating that for every $\mathcal{Q}$-morphism α, the $\mathcal{Q}$-morphism $\dot{\gamma}$ associated with $\gamma = \widetilde{\alpha}$, the closure operator associated with α, is the coequalizer of the congruence of α. This is used then, together with Proposition 7.7, to prove that all epis in $\mathcal{Q}$-Mod are regular. As a corollary we obtain that $\mathcal{Q}$-Mod has the strong amalgamation property.

In case $\mathcal{Q}$ is small, we prove that $\mathcal{Q}$-Mod has a separating set, and therefore it is wellpowered, and cowellpowered by duality. The cardinality of the separating set is the cardinality of $\mathcal{Q}$, and therefore in the case that $\mathcal{Q}$ is a quantale, that is, $\mathcal{Q}$ only has one object, we obtain that $\mathcal{Q}$-Mod has a separator. As corollaries we obtain the strong completeness and strong cocompleteness of $\mathcal{Q}$-Mod, and that it is $(\mathrm{Epi}, \mathrm{Mono})$-structured.

Proposition 8.1. *If $\mathcal{Q}$ is a quantaloid, then the category $\mathcal{Q}$-Mod has arbitrary products and coproducts. Moreover, $\mathcal{Q}$-Mod has biproducts and a zero object.*

Proof. Let $\mathbf{1}$ be the unique (up to isomorphism) complete lattice with just one element. It is a zero object in the category $\mathcal{S}\ell$, since for every $\mathbf{R} \in \mathcal{S}\ell$, the unique residuated map $¡_{\mathbf{R}} : \mathbf{1} \to \mathbf{R}$ is the map that sends the element in $\mathbf{1}$ to $\perp_{\mathbf{R}}$, with residuum the unique map from $\mathbf{R}$ to $\mathbf{1}$, $!_{\mathbf{R}} : \mathbf{R} \to \mathbf{1}$. Moreover, this map is residuated with residuum $!^{+}_{\mathbf{R}} : 1 \to R$ the map that sends the element in 1 to $\top^{\mathbf{R}}$.

Let $\mathcal{Q}$ be a quantaloid, and define $\mathit{1} : \mathcal{Q} \to \mathcal{S}\ell$ as the constant functor with value $\mathbf{1}$. First note that $\mathit{1}$ is an enriched functor, since for every $A, B \in \mathcal{Q}$, the constant map $[A, B] \to [\mathbf{1}, \mathbf{1}] \cong \mathbf{1}$ is residuated. Let us see that $\mathit{1}$ is a zero object in $\mathcal{Q}$-Mod.

If $T \in \mathcal{Q}$-Mod, then for every $A \in \mathcal{Q}$, $!_{TA} : TA \to \mathbf{1}$ is the unique residuated map from TA to $\mathbf{1}$, and then $!_T : T \to \mathit{1}$ is a natural transformation, whose naturality follows from the fact that $\mathbf{1}$ is terminal in $\mathcal{S}\ell$. Analogously, there exists a unique natural transformation $¡_T : \mathit{1} \to T$, whose components are $¡_{TA} : \mathbf{1} \to TA$.

Suppose now that $\{T_i : i \in I\}$ is a nonempty family of $\mathcal{Q}$-modules. Since for every $i \in I$, and every $A \in \mathcal{Q}$, T_iA is a complete lattice, then $\prod_I T_iA$ is a complete lattice, where the joins are computed componentwise. Let $\prod_I T_i : \mathcal{Q} \to \mathcal{S}\ell$ be the functor determined by

$$\begin{tikzcd} A \arrow[r, mapsto] \arrow[d, "a"'] & \prod_I T_iA \arrow[d, "\prod_I T_i(a)"] \\ B \arrow[r, mapsto] & \prod_I T_iB \end{tikzcd}$$

where, for every $x \in \prod_I T_iA$, $\big(\prod_I T_i(a)\big)x = \big(T_i(a)x_i\big)_{i\in I}$. It is straightforward to check that it is an enriched functor. The projections $\pi^j : \prod_I T_i \to T_j$ are the natural maps that in each component have the corresponding projection: for every $A \in \mathcal{Q}$, $\pi^j_A : \prod_i T_iA \to T_jA$. They are residuated maps with residuum $(\pi^j_A)^+ : T_jA \to \prod_I T_iA$ determined by

$$\big((\pi^j_A)^+(y)\big)_i = \begin{cases} \top^{T_iA} & \text{if } i \neq j, \\ y & \text{if } i = j. \end{cases}$$

In order to see that the family $\{T_i : i \in I\}$ also has a coproduct, note that the family $\{T_i^\partial : i \in I\}$ has a product in $\mathcal{Q}^{\mathrm{op}}$-$Mod$, and thence $\big(\prod_I (T_i^\partial)\big)^\partial$ is a coproduct of $\{T_i : i \in I\}$. It is straightforward to see that, for every $A \in \mathcal{Q}$, $\big(\prod_I (T_i^\partial)\big)^\partial A = \prod_I T_iA$, and analogously for arrows, whence we obtain that $\big(\prod_I (T_i^\partial)\big)^\partial = \prod_I T_i$. The inclusions $e^j : T_j \to \prod_I T_i$ are the duals of the projections, that is, for every $A \in \mathcal{Q}$, and every $x \in T_jA$,

$$\big(e^j_A(x)\big)_i = \begin{cases} \top^{T_i^\partial A} = \perp_{T_iA} & \text{if } i \neq j, \\ x & \text{if } i = j. \end{cases}$$

Thus, we have proved that $\mathcal{Q}$-$\mathcal{M}od$ has arbitrary biproducts. $\square$

Proposition 8.2. *If $\mathcal{Q}$ is a quantaloid, then the category $\mathcal{Q}$-$\mathcal{M}od$ has equalizers and coequalizers.*

Proof. Suppose that $\alpha^1, \alpha^2 : T \to T'$ are two $\mathcal{Q}$-morphisms, and let E be the $\mathcal{Q}$-submodule of T determined by: for every $A \in \mathcal{Q}$, $|EA| = \{x \in TA : \alpha^1 x = \alpha^2 x\}$. In order to prove that E is well defined, suppose that $\{x_i : i \in I\} \subseteq |EA|$. Then, we have that $\alpha^1(\bigvee x_i) = \bigvee \alpha^1 x_i = \bigvee \alpha^2 x_i = \alpha^2(\bigvee x_i)$. Let suppose furthermore that $x \in |EA|$ and $a : A \to B$ is in $\mathcal{Q}$. Then, $\alpha^1_B(a *_T x) = a *_{T'} \alpha^1_A x = a *_{T'} \alpha^2_A x = \alpha^2_B(a *_T x)$. Thus, in virtue of Lemma 4.7, E is a $\mathcal{Q}$-submodule of T. Let $e : E \hookrightarrow T$ be the inclusion $\mathcal{Q}$-morphism. It is evident that $\alpha^1 e = \alpha^2 e$, and that for every $\eta : H \to T$ such that $\alpha^1 \eta = \alpha^2 \eta$, the image of η is a $\mathcal{Q}$-submodule of E. Hence, η factorizes through e uniquely. Thus, $e : E \to T$ is the equalizer of α^1 and α^2. $\square$

The following result is fundamental in order to understand the categories of modules over a quantaloid. Some of its consequences are that every epi is regular in $\mathcal{Q}$-$\mathcal{M}od$, the amalgamation property for $\mathcal{Q}$-$\mathcal{M}od$, and the (Epi, Mono)-structure of $\mathcal{Q}$-$\mathcal{M}od$, in the case $\mathcal{Q}$ is small.

Proposition 8.3. *Let $\alpha : T \to T'$ be a $\mathcal{Q}$-morphism, (η^1, η^2) the congruence of α, and $\gamma = \widetilde{\alpha}$ the closure operator associated with α. Then $\dot{\gamma} : T \to T_\gamma$ is the coequalizer of $\eta^1, \eta^2 : H \to T$.*

Proof. The congruence of α is the pair $\eta^1, \eta^2 : H \to T$ such that the following diagram is a pullback:

$$\begin{tikzcd} H \arrow[r, "\eta^1"] \arrow[d, "\eta^2"'] & T \arrow[d, "\alpha"] \\ T \arrow[r, "\alpha"] & T' \end{tikzcd}$$

Its existence is guaranteed, since we have proved that the category $\mathcal{Q}$-$\mathcal{M}od$ has arbitrary products and equalizers, and hence it is complete. Moreover, the $\mathcal{Q}$-module H is taken to be the $\mathcal{Q}$-submodule of $T \times T$ such that, for every $A \in \mathcal{Q}$, $HA = \{\langle x, y\rangle : \alpha x = \alpha y\}$, and $\eta^i : H \to T$ is the composition of the inclusion $H \hookrightarrow T \times T$ with the projection $\pi^i : T \times T \twoheadrightarrow T$.

If $\varepsilon : T \twoheadrightarrow Q$ is the coequalizer of η^1, η^2, then by duality, $\varepsilon^\partial : Q^\partial \to T^\partial$ is the equalizer of $(\eta^1)^\partial$ and $(\eta^2)^\partial$. We know that this is the inclusion $\varepsilon^\partial : E \hookrightarrow T^\partial$ where for every $A \in \mathcal{Q}$, $EA = \{x \in TA : (\eta^1)^\partial_A x = (\eta^2)^\partial_A x\} = \{x \in TA : (\eta^1_A)^+ x = (\eta^2_A)^+ x\} = \{x \in TA : \langle x, \gamma_A x\rangle = \langle \gamma_A x, x\rangle\} = \{x \in TA : x = \gamma_A x\} = T_\gamma A$. Thus, $Q = E^\partial = T_\gamma$ and $\dot{\gamma} = \varepsilon^\partial : T \twoheadrightarrow T_\gamma$ is the coequalizer of η^1 and η^2. $\square$

Corollary 8.4. *Let $\mathcal{Q}$ be a quantaloid. Then the classes* RegEpi, ExtrEpi *and* Epi *coincide in $\mathcal{Q}$-$\mathcal{M}od$. Dually, the classes* RegMono, ExtrMono *and* Mono *also coincide in $\mathcal{Q}$-$\mathcal{M}od$.*

Proof. In virtue of Proposition 7.7, if β is an epi and $\gamma = \widetilde{\beta}$ is its associated closure operator, then β is isomorphic to $\dot{\gamma}$, which is a regular epi, by Proposition 8.3. We have then that Epi $\subseteq$ RegEpi, which together with the general inclusions RegEpi $\subseteq$ ExtrEpi $\subseteq$ Epi proves the result. $\square$

We will prove in what follows the strong amalgamation property for the categories of modules over quantaloids. As a previous result, we study in more detail how are the pullbacks of epis in general in $\mathcal{Q}$-$\mathcal{M}od$, and prove in the following lemma that epis are preserved by pullbacks.

Lemma 8.5. *Let $\mathcal{Q}$ be a quantaloid, and the following*

$$\begin{tikzcd} H \arrow[r] \arrow[d, "\overline{\varepsilon}"'] & T_1 \arrow[d, "\varepsilon"] \\ T_2 \arrow[r, "\theta"] & R \end{tikzcd}$$

a pullback in the category $\mathcal{Q}$-$\mathcal{M}od$. In this situation, if ε is epi then $\overline{\varepsilon}$ also is epi.

Proof. We can suppose that H is the standard pullback of θ and ε, which is determined by its universe in every component by $|HA| = \{\langle x, y\rangle \in T_1A \times T_2A : \varepsilon_A x = \theta_A y\}$, and $\overline{\varepsilon}$ is the composition of the inclusion $H \hookrightarrow T_1 \times T_2$ with the projection $\pi^2 : T_1 \times T_2 \to T_2$. Suppose that $A \in \mathcal{Q}$ and $y \in T_2A$. Thus, if $\varepsilon : T_1 \twoheadrightarrow R$ is an epi, then there exists $x \in T_1A$ such that $\varepsilon_A x = \theta_A y$. That is, there exists $z = \langle x, y\rangle \in HA$ such that $\overline{\varepsilon}_A z = y$, whence we obtain that $\overline{\varepsilon}_A$ is onto for every A. Therefore, $\overline{\varepsilon}$ is epi. $\square$

Recall that a diagram in a category $\mathcal{C}$

$$\begin{tikzcd} A \arrow[r, "n"] \arrow[d, "m"'] & C \arrow[d, "u"] \\ B \arrow[r, "v"] & D \end{tikzcd} \tag{5}$$

is an *amalgamation* (of m and n) if it is a pushout and all the arrows are monos. An amalgamation is *strong* if moreover it is a pullback. A pair of monos $m : A \to B$ and $n : A \to C$, can be (*strongly*) *amalgamated* if a (strong) amalgamation (5) exists. A category $\mathcal{C}$ has the (*strong*) *amalgamation property* if every pair of monos with the same domain can be (strongly) amalgamated.

Corollary 8.6. *Let $\mathcal{Q}$ be a quantaloid. Then the category $\mathcal{Q}$-$\mathcal{M}od$ has the strong amalgamation property.*

Proof. Suppose that $\eta : R \to T_1$ and $\nu : R \to T_2$ is a pair of monos in $\mathcal{Q}$-$\mathcal{M}od$. Then, their duals $\eta^\partial : (T_1)^\partial \twoheadrightarrow R^\partial$ and $\nu : (T_2)^\partial \twoheadrightarrow R^\partial$ constitute a pair of epis in $\mathcal{Q}^{\mathrm{op}}$-$\mathcal{M}od$. Since this category is complete, their pullback exists and is represented by the first of the two diagrams:

$$\begin{tikzcd} H \arrow[r, two heads, "\theta"] \arrow[d, two heads, "\chi"'] & (T_1)^\partial \arrow[d, two heads, "\nu^\partial"] \\ (T_2)^\partial \arrow[r, two heads, "\eta^\partial"] & R^\partial \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "\nu"] \arrow[d, "\eta"'] & T_1 \arrow[d, "\theta^\partial"] \\ T_2 \arrow[r, "\chi^\partial"] & H^\partial \end{tikzcd}$$

In virtue of the preceding lemma, both χ and θ are epis, and then their duals are monos. Therefore, the second diagram, which is the dual of the first one, is an amalgamation of ν and η. Hence, we have proved that the category $\mathcal{Q}$-$\mathcal{M}od$ has the amalgamation property. In order to end this proof, note that in virtue of Theorem 1 of [Tho82], the strong amalgamation property in a cocomplete category is equivalent to the equation Mono = RegMono, which is true for $\mathcal{Q}$-$\mathcal{M}od$, by Corollary 8.4. $\square$

Proposition 8.7. *Let $\mathcal{Q}$ be a small quantaloid. Then the set $\{h_A : A \in \mathcal{Q}\}$ is a separating set for the category $\mathcal{Q}$-$\mathcal{M}od$.*

Proof. Suppose that $\alpha, \beta : T \to T'$ are two different $\mathcal{Q}$-morphisms, and let $A \in \mathcal{Q}$ and $x \in TA$ such that $\alpha_A x \neq \beta_A x$. Then, it is easy to see evaluating at 1_A that $\alpha \cdot \mu^x \neq \beta \cdot \mu^x$, where $\mu^x : h_A \to T$ is the unique morphisms in $[h_A, T]$ corresponding to $x \in TA$ by the Yoneda Lemma. $\square$

Remark 8.8. Note that, in the particular case when $\mathcal{Q}$ has just one object $\star$, that is, $\mathcal{Q}$ is a the quantaloid associated with a quantale $\mathcal{A} = \langle A, \cdot, 1\rangle$, we have that $\{h_\star\}$ is a separating set for $\mathcal{Q}$-$\mathcal{M}od \cong \mathcal{A}$-$\mathcal{M}od$, that is, $h_\star$ is a separator for $\mathcal{Q}$-$\mathcal{M}od$. It is easy to see that this corresponds exactly with the $\mathcal{A}$-module $\mathbb{A} = \langle A, \cdot\rangle$.

Corollary 8.9. *Let $\mathcal{Q}$ be a small quantaloid. Then the category $\mathcal{Q}$-$\mathcal{M}od$ is wellpowered and also cowellpowered.*

Proof. This is a consequence of the precedent proposition, since every category with a separating set is concretizable over $\mathcal{S}et$ (see Exercise 7Q of [AHS06]), and every construct is wellpowered and cowellpowered (see 7.88 of [AHS06]). $\square$

Corollary 8.10. *Let $\mathcal{Q}$ be a small quantaloid. Then the category $\mathcal{Q}$-Mod is strongly complete and strongly cocomplete.*

Proof. We know that $\mathcal{Q}$-Mod is complete, since it has products and equalizers. Moreover it is strongly complete, since it is wellpowered. Finally, it is strongly cocomplete by duality. $\square$

Corollary 8.11. *Let $\mathcal{Q}$ a small quantaloid. Then the category $\mathcal{Q}$-Mod is* $(\mathrm{Epi}, \mathrm{Mono})$*-structured.*

Proof. Since the category $\mathcal{Q}$-Mod is strongly complete in virtue of the precedent corollary, it is $(\mathrm{ExtrEpi}, \mathrm{Mono})$-structured. But the classes ExtrEpi and Epi coincide in $\mathcal{Q}$-Mod, what proves the result. $\square$

9 Interpretability and representability in $\mathcal{Q}$-Mod

Definition 9.1. If $\mathcal{Q}$ is a quantaloid, T and T' are two $\mathcal{Q}$-modules, and γ and δ are closure operators on T and T', respectively, then a *semi-interpretation* of γ into δ is a $\mathcal{Q}$-morphism $\tau : T \to T'$ such that for every $A \in \mathcal{Q}$, and every $x, x' \in TA$,

$$x \leqslant \gamma_A(x') \;\Rightarrow\; \tau_A x \leqslant \delta_A\big(\tau_A(x')\big).$$

τ is an *interpretation* of γ into δ if for every $A \in \mathcal{Q}$, and every $x, x' \in TA$,

$$x \leqslant \gamma_A(x') \;\Leftrightarrow\; \tau_A x \leqslant \delta_A\big(\tau_A(x')\big).$$

In case τ is a (semi-)interpretation of γ into δ, we say that γ is *(semi-)interpretable* into δ by τ.

The following Lemma establishes characterizations for semi-interpretability and interpretability in terms of the order of the lattice of closure operators. It follows immediately from the definitions and Proposition 7.15.

Lemma 9.2. *If $\tau : T \to T'$ is a $\mathcal{Q}$-morphism between two $\mathcal{Q}$-modules, and γ and δ are closure operators on T and T', respectively, then*

(i) γ is semi-interpretable into δ by τ if and only if $\gamma \leqslant \delta^\tau$;
(ii) γ is interpretable into δ by τ if and only if $\gamma = \delta^\tau$. $\square$

We have proved that for every $\mathcal{Q}$-morphism $\tau : T \to T'$, the map between the lattices of closure operators on T and T', ${}^\tau(\_) : \mathbf{Clop}(T) \to \mathbf{Clop}(T')$ is a residuated map with residuum $(\_)^\tau$. Therefore, their composition $\widehat{\tau} = ({}^\tau(\_))^\tau : \mathrm{Clop}(T) \to \mathrm{Clop}(T)$ is a closure operator on $\mathbf{Clop}(T)$. The following result is easily proved taking into account this fact and the reformulation of the notions of semi-interpretability and interpretability of the previous Lemma.

Corollary 9.3. *If $\tau : T \to T'$ is a $\mathcal{Q}$-morphism between two $\mathcal{Q}$-modules, then*

(i) every closure operator γ on T is semi-interpretable in ${}^\tau\gamma$;
(ii) the closure system associated with $\widehat{\tau}$ is $\big\{\delta^\tau : \delta \in \mathrm{Clop}(T')\big\}$; therefore, a closure operator on T is interpretable by τ if and only if it is $\widehat{\tau}$-closed;
(iii) a closure operator γ on T is interpretable by τ if and only if it is interpretable in ${}^\tau\gamma$ by τ.

Proof.

(i) This is a consequence of the expansiveness of $\widehat{\tau}$, since $\gamma \leqslant \widehat{\tau}(\gamma) = ({}^\tau\gamma)^\tau$.
(ii) The first part is a general result for residuated maps, and the second part follows from the first and the characterization of Lemma 9.2.
(iii) If γ is interpretable by τ, then by (ii), it is $\widehat{\tau}$-closed. Therefore, $\gamma = \widehat{\tau}(\gamma) = ({}^\tau\gamma)^\tau$. The other implication is obvious. $\square$

In view of the previous corollary, we can give another characterization of interpretable closure operators by a fixed $\mathcal{Q}$-morphism. Indeed, it is a description of the closure system associated with $\widehat{\tau}$ as a principal filter of $\mathbf{Clop}(T)$.

Proposition 9.4. *Let $\mathcal{Q}$ be a quantaloid, and $\tau : T \to T'$ a $\mathcal{Q}$-morphism between two $\mathcal{Q}$-modules. Then the set of closure operators on T that are interpretable by τ is a principal filter of $\mathbf{Clop}(T)$ generated by $\widetilde{\tau}$, the closure operator on T determined by τ.* □

Proof. Let γ be a closure operator on T. Then for every $A \in \mathcal{Q}$, the closure system associated with $\widehat{\tau}(\gamma)_A$ is the set

$$\begin{aligned} \mathrm{Cl}\big(({}^{\tau}\gamma)^{\tau}_A\big) &= \{\tau^{+}_A y : y \in T'_{({}^{\tau}\gamma)}A\} = \{\tau^{+}_A y : y \in T'A,\ \tau^{+}_A y \in T_\gamma A\} \\ &= \{\tau^{+}_A y : y \in T'A\} \cap T_\gamma A = \mathrm{Cl}(\widetilde{\tau}_A) \cap T_\gamma A. \end{aligned}$$

Thus, we have that

$$\begin{aligned} \gamma \text{ is interpretable by } \tau \ &\Leftrightarrow\ \gamma \text{ is } \widehat{\tau}\text{-closed} \ \Leftrightarrow\ \gamma = \widehat{\tau}(\gamma) \\ &\Leftrightarrow\ \text{for every } A \in \mathcal{Q},\ T_\gamma A = \mathrm{Cl}\big(({}^{\tau}\gamma)^{\tau}_A\big) = \mathrm{Cl}(\widetilde{\tau}_A) \cap T_\gamma A \\ &\Leftrightarrow\ \text{for every } A \in \mathcal{Q},\ T_\gamma A \subseteq \mathrm{Cl}(\widetilde{\tau}_A) \\ &\Leftrightarrow\ \widetilde{\tau} \leqslant \gamma. \end{aligned}$$

□

Corollary 9.5. *Let $\mathcal{Q}$ be a quantaloid and $\tau : T \to T'$ and $\mathcal{Q}$-morphism. If a closure operator γ on T is interpretable by τ, then every extension of γ is interpretable by τ.* □

Note that this is a generalization of Theorem 2.15 of [BR03], which says that if a sentential logic has an algebraic semantics, so does every of its extensions extensions, and that we have arrived to it using mainly the residuation property of ${}^{\tau}(\_)$.

Definition 9.6. If $\mathcal{Q}$ is a quantaloid, T and T' are two $\mathcal{Q}$-modules, and γ and δ are closure operators on T and T', respectively, then a *semi-representation* of γ into δ is a $\mathcal{Q}$-morphism $\alpha : T_\gamma \to T'_\delta$. A semi-representation α of γ into δ is *induced* if there exists a morphism $\tau : T \to T'$ such that the following diagram commutes:

$$\begin{tikzcd} T \arrow[r, "\tau"] \arrow[d, two heads, "\dot{\gamma}"'] & T' \arrow[d, two heads, "\dot{\delta}"] \\ T_\gamma \arrow[r, "\alpha"] & T'_\delta \end{tikzcd}$$

In that case, we say that τ *induces* α. A *representation* of γ into δ is a semi-representation which is a mono. Furthermore, we say that γ is *(semi-)representable* into δ if there exists a (semi-)representation of γ into δ, and that it is *(semi-)representable* by τ if τ induces a (semi-)representation of γ into δ.

Remark 9.7. If α is a semi-interpretation of γ into δ induced by τ, then α is completely determined by δ, γ, and τ. Indeed, $\alpha = \dot{\delta}\tau\restriction_{T_\gamma}$.

The following proposition establishes that (semi-)interpretability implies induced (semi-)representability.

Proposition 9.8. *Let $\mathcal{Q}$ be a quantaloid, $\tau : T \to T'$ a $\mathcal{Q}$-morphism between to $\mathcal{Q}$-modules, γ and δ closure operators on T and T', respectively and $\alpha = \dot{\delta}\tau\restriction_{T_\gamma}$. Then,*

(i) if γ is semi-interpretable into δ by τ, then α is the unique semi-representation of γ into δ induced by τ;

(ii) if γ is interpretable into δ by τ, then α is the unique representation of γ into δ induced by τ.

Proof.

(i) Let us prove that $\alpha : T_\gamma \to T'_\delta$ is a $\mathcal{Q}$-morphism. First, note that for every $A \in \mathcal{Q}$, $\alpha_A = \dot{\delta}_A\tau_A{\restriction}_{T_\gamma} : T_\gamma A \to T'_\delta A$ is residuated. Its residuum can be easily calculated: for every $x \in T_\gamma A$ and every $z \in T'_\delta A$,

$$\begin{array}{lll} \delta_A\tau_A x \leqslant z & & \text{in } T'_\delta A \\ \quad\Leftrightarrow\ \tau_A x \leqslant z & & \text{in } T'A, \text{ because } z \in T'_\delta A, \\ \quad\Leftrightarrow\ x \leqslant \tau_A^+ z & & \text{in } TA, \text{ because of the residuation property of } \tau_A, \\ \quad\Leftrightarrow\ x \leqslant \tau_A^+ z & & \text{in } T_\gamma A, \text{ because } z \in T'_\delta A \ \Rightarrow\ \tau_A^+ z \in T_{\delta^\tau}A \subseteq T_\gamma A, \text{ since } \gamma \leqslant \delta^\tau. \end{array}$$

Thus, the restriction of τ_A^+ to a map $T'_\delta A \to T_\gamma A$ is the residuum of α_A. In order to prove the structurality of α, first note that $\gamma \leqslant \delta^\tau = \tau^\partial\delta\tau \ \Rightarrow\ \tau\gamma \leqslant \delta\tau \ \Rightarrow\ \delta\tau\gamma \leqslant \delta\tau \leqslant \delta\tau\gamma \ \Rightarrow\ \delta\tau = \delta\tau\gamma$. Suppose now that $a : A \to B$ is in $\mathcal{Q}$ and $x \in T_\gamma A$. Then,

$$\begin{aligned} a *_{T'_\delta} \alpha_A x &= \delta_B(a *_{T'} \delta_A\tau_A x) = \delta_B(a *_{T'} \tau_A x) = \delta_B\tau_B(a *_T x) = \delta_B\tau_B\gamma_B(a *_T x) \\ &= \alpha_B(a *_{T_\gamma} x). \end{aligned}$$

(ii) The only thing we have to prove, in virtue of (i), is that if τ is an interpretation of γ into δ, then for every $A \in \mathcal{Q}$, α_A is injective. Since $\alpha_A : T_\gamma A \to T'_\delta A$ is a residuated map, it is injective if and only if its residuum is its left inverse. Note that, by hypothesis, $\gamma_A = \delta_A^\tau = \tau_A^+\delta_A\tau_A$. Hence, the correspondent restrictions are:

$$id_{T_\gamma A} = \gamma_A{\restriction}_{T_\gamma A} = \tau_A^+{\restriction}_{T_\delta A}(\delta_A\tau_A){\restriction}_{T_\gamma A} = \alpha_A^+\alpha_A.$$

In both cases, the uniqueness of α is evident, by the previous remark. □

Proposition 9.9. *Let $\mathcal{Q}$ be a quantaloid, T and T' two $\mathcal{Q}$-modules, $\tau : T \to T'$ a $\mathcal{Q}$-morphism, and γ and δ two closure operators on T and T', respectively. Then,*

(i) if τ induces a semi-representation of γ into δ, then τ is a semi-interpretation of γ into δ;

(ii) if τ induces a representation of γ into δ, then τ is an interpretation of γ into δ.

Proof.

(i) Suppose that α is a semi-representation of γ into δ induced by τ. Then, $\dot{\delta}\tau = \alpha\dot{\gamma}$, which implies that for every $A \in \mathcal{Q}$, $\dot{\delta}_A\tau_A\gamma_A = \alpha_A\dot{\gamma}_A\gamma_A = \alpha_A\dot{\gamma}_A = \dot{\delta}_A\tau_A$. Composing with the inclusion $j_A : T'_\delta A \to T'A$, we obtain $\delta_A\tau_A\gamma_A = \delta_A\tau_A$. Therefore, $\tau_A\gamma_A \leqslant \delta_A\tau_A$, and by the residuation property of τ_A, we obtain $\gamma_A \leqslant \tau_A^+\delta_A\tau_A = \delta_A^\tau$.

(ii) Suppose now that α is a representation of γ into δ induced by τ. Then, taking duals in $\alpha\dot{\gamma} = \dot{\delta}\tau$, we obtain $i\alpha^\partial = \dot{\gamma}^\partial\alpha^\partial = (\alpha\dot{\gamma})^\partial = (\dot{\delta}\tau)^\partial = \tau^\partial\dot{\delta}^\partial = \tau^\partial j$, where $i : (T_\gamma)^\partial \to T^\partial$ and $j : (T'_\delta)^\partial \to (T')^\partial$ are the inclusions. Thus, for every $A \in \mathcal{Q}$, $\dot{\gamma}_A\tau_A^+ j_A = \dot{\gamma}_A i_A\alpha_A^+ = \alpha_A^+$. We know that, for every $A \in \mathcal{Q}$, α_A^+ is a left inverse of α_A, since α_A is injective. Hence, from the following equalities

$$\dot{\gamma}_A\delta_A^\tau = \dot{\gamma}_A\tau_A^+\delta_A\tau_A = (\dot{\gamma}_A\tau_A^+ j_A)(\dot{\delta}_A\tau_A) = \alpha_A^+\alpha_A\dot{\gamma}_A = \dot{\gamma}_A,$$

it follows that $\gamma_A\delta_A^\tau = \gamma_A$, and therefore $\delta_A^\tau \leqslant \gamma_A$. The other inequality follows from (i). □

As it is expected, whereas interpretability implies representability, as we have shown in Proposition 9.8, the reverse implication is not true, in general. That is equivalent, by Proposition 9.9, to saying that representations are not always induced. The following theorem characterizes the modules such that all the representations of closure operators on them are induced.

Recall that if $\mathcal{Q}$ is a quantaloid, a $\mathcal{Q}$-module P is *projective* if for every epi $\beta : T \twoheadrightarrow T'$ and every $\mathcal{Q}$-morphism $\alpha : P \to T'$, there exists a $\mathcal{Q}$-morphism $\overline{\alpha} : P \to T$ completing the diagram:

$$\begin{tikzcd} & T \arrow[d, two heads, "\beta"] \\ P \arrow[ur, dashed, "\exists\,\overline{\alpha}"] \arrow[r, "\alpha"'] & T' \end{tikzcd} \qquad (6)$$

Theorem 9.10. *If $\mathcal{Q}$ is a quantaloid, then a $\mathcal{Q}$-module P is projective if and only if every representation of a closure operator on P into another closure operator is induced.*

Proof. If P is projective, γ is a closure operator on P, δ is a closure operator on T and $\alpha : P_\gamma \to T_\delta$ is a representation, then by the projectivity of P, the following diagram can be completed:

$$\begin{tikzcd}
P \arrow[r, dashed] \arrow[d, two heads, "\dot{\gamma}"'] \arrow[dr] & T \arrow[d, two heads, "\dot{\delta}"] \\
P_\gamma \arrow[r, "\alpha"] & T_\delta
\end{tikzcd}$$

which implies that α is induced.

Suppose now that $\beta : T \twoheadrightarrow T'$ is an epi, and $\alpha : P \to T'$ is a $\mathcal{Q}$-morphism. Let $\delta = \widetilde{\beta}$, the closure operator on T determined by β. Hence, there exists an isomorphism $\eta : T' \cong T_\delta$. Let γ be the closure operator on P such that $\eta \cdot \alpha$ factorizes trough P_γ, that is $\gamma = \widetilde{\eta \cdot \alpha}$. Hence, we have the commutativity of the solid part of the diagram

$$\begin{tikzcd}
P \arrow[rr, dashed, "\tau"] \arrow[dr, "\alpha"] \arrow[dd, two heads, "\dot{\gamma}"'] & & T \arrow[dl, two heads, "\beta"'] \arrow[dd, two heads, "\dot{\delta}"] \\
& T' \arrow[dr, "\eta", "\cong"'] & \\
P_\gamma \arrow[rr, tail, "\zeta"] & & T_\delta
\end{tikzcd}$$

Then, the square can be completed with a $\mathcal{Q}$-morphism τ, since ζ is a representation of γ into δ. Thus, $\beta\tau = \eta^{-1}\eta\beta\tau = \eta^{-1}\dot{\delta}\tau = \eta^{-1}\zeta\dot{\gamma} = \eta^{-1}\eta\alpha = \alpha$, as we wanted to prove. ☐

Remark 9.11. Note that the injectivity of α is not used in the first part of the proof of the preceding theorem, and also that every representation is a semi-representation. Therefore, we also have that a $\mathcal{Q}$-module P is projective if and only if every semi-representation of a closure operator on P into another closure operator is induced.

10 Cyclic $\mathcal{Q}$-modules and cyclic projective $\mathcal{Q}$-modules

Definition 10.1. If $\mathcal{Q}$ is a quantaloid and T is a $\mathcal{Q}$-module, then T is said to be *cyclic* if there exist an object $A \in \mathcal{Q}$, and an element $v \in TA$, such that for every $B \in \mathcal{Q}$, and every $x \in TB$, there exists an arrow $v_{B,x} : A \to B$ in $\mathcal{Q}$ such that $v_{B,x} *_T v = x$. The pair $\langle A, v\rangle$ is called a *generator* of T.

In what follows, if the object B is clear from the context, we suppress the subindex B of $v_{B,x} : A \to B$, and denote it just by $v_x : A \to B$. The following lemma is a characterization of the property of being a generator.

Lemma 10.2. *Let $\mathcal{Q}$ be a quantaloid, T a $\mathcal{Q}$-module, $A \in \mathcal{Q}$, and $v \in TA$. Then $\langle A, v\rangle$ is a generator of T if and only if the $\mathcal{Q}$-morphism $\mu^v : h_A \to T$, given by the Yoneda Lemma, is an epi.*

Proof. Recall that epis in $\mathcal{Q}$-Mod are those $\mathcal{Q}$-morphisms whose components are onto. Therefore, this is just a reformulation of the definition of a generator, since for every $A, B \in \mathcal{Q}$, $h_A B = [A, B]$, and for every $v \in TA$, $\mu^v = \_ *_T v$. ☐

The following proposition characterizes the cyclics of $\mathcal{Q}$-Mod as the quotients of the modules of the form h_A. In the case $\mathcal{Q}$ is a quantale, that is, $\mathcal{Q}$ has just one element, this proposition coincides with Lemma 5.4 of [GT09].

Proposition 10.3. *Let $\mathcal{Q}$ be a quantaloid, T a $\mathcal{Q}$-module, and $A \in \mathcal{Q}$. Then T is cyclic with generator $\langle A, v\rangle$ if and only if T is isomorphic to a quotient of h_A.*

Proof. Suppose that T is cyclic and $\langle A, v\rangle$ is a generator of T. Consider the $\mathcal{Q}$-morphism $\mu^v : h_A \to T$ given by the Yoneda Lemma. By Lemma 10.2, $\mu^v : h_A \twoheadrightarrow T$ is an epi in $\mathcal{Q}$-$\mathcal{M}od$. In virtue of Proposition 7.7, μ^v is isomorphic to $\dot{\gamma}$, where $\gamma = \widetilde{\mu^v}$ is the closure operator on h_A associated with μ^v. That is, there exists an iso $\eta : (h_A)_\gamma \cong T$ rendering commutative the diagram

$$\begin{tikzcd} & h_A \arrow[dl, two heads, "\dot{\gamma}"'] \arrow[dr, two heads, "\mu^v"] & \\ (h_A)_\gamma \arrow[rr, "\eta", "\cong"'] & & T \end{tikzcd}$$

In the other direction, suppose that T is isomorphic to a quotient of h_A, that is, there exists an epi $\varepsilon : h_A \twoheadrightarrow T'$ and an iso $\eta : T' \cong T$. By Yoneda Lemma, there exists $v \in TA$ (indeed, $v = (\eta \cdot \varepsilon)(1_A)$) such that $\mu^v = \eta \cdot \varepsilon : h_A \twoheadrightarrow T$, which is an epi, since η is so. Hence, in virtue of Lemma 10.2, $\langle A, v\rangle$ is a generator of T. $\square$

The following is a technical lemma, establishing that if $\langle A, v\rangle$ is a generator of a $\mathcal{Q}$-module T, then for every $B \in \mathcal{Q}$ and every $x \in TB$, we can choose v_x as $(x/_T v)$. This will be used later on.

Lemma 10.4. *If $\mathcal{Q}$ is a quantaloid and T is a $\mathcal{Q}$-module, then $\langle A, v\rangle$ is a generator of T if and only if for every $B \in \mathcal{Q}$, and for every $x \in TB$, $(x/_T v) *_T v = x$.*

Proof. Let $\langle A, v\rangle$ be a generator of T, and for every $B \in \mathcal{Q}$, let $v_x \in [A, B]$, such that $v_x *_T v = x$. Then, $v_x \leqslant x/_T v$, and therefore

$$x = v_x *_T v \leqslant (x/_T v) *_T v \leqslant x,$$

whence $(x/_T v) *_T v = x$. The other direction is evident, taking $v_x = x/_T v$. $\square$

Definition 10.5. If $\mathcal{Q}$ is a quantaloid and T is a cyclic $\mathcal{Q}$-module, then a *g-variable* for T is a generator $\langle A, v\rangle$ such that there exists an arrow $u : A \to A$ in $\mathcal{Q}$ satisfying the following two properties:

1. $u *_T v = v$,
2. for every $a : A \to B$, $\big((a *_T v)/_T v\big) \circ u = a \circ u$.

Remark 10.6. Observe that when the notion of g-variable for $\mathcal{Q}$-modules specializes to an M-set $\langle X, \cdot\rangle$, what we obtain is a set $v \subseteq X$ such that there exists $u \subseteq M$ satisfying the properties 1., 2., and 3. of Proposition 2.3. It was proved in Theorem 2.10 that, in that case, any element of v is indeed g-variable for the M-set $\langle X, \cdot\rangle$. On the other direction, if p is a g-variable for $\langle X, \cdot\rangle$, then $\{p\}$ is a g-variable for the module associated with $\langle X, \cdot\rangle$.

Given a morphism $u : A \to A$ in $\mathcal{Q}$, we can define the $\mathcal{Q}$-module $h_A u$ as the image of $\mu^u : h_A \to h_A$. Thus, for every $B \in \mathcal{Q}$, $(h_A u)B = \{b \circ u : b \in h_A B\}$. The next proposition is a characterization of the modules having a g-variable as those of the form $h_A u$, where $u : A \to A$ is an idempotent, up to isomorphism.

Proposition 10.7. *A $\mathcal{Q}$-module T has a g-variable if and only if there exists an idempotent $u : A \to A$ for some $A \in \mathcal{Q}$ such that $T \cong h_A u$.*

Proof. Suppose that $\langle A, v\rangle$ is a g-variable for a $\mathcal{Q}$-module T. Then, in virtue of 1. and 2. of Definition 10.5:

$$u \circ u = \big((u *_T v)/_T v\big) \circ u = (v/_T v) \circ u = \big((1_A *_T v)/_T v\big) \circ u = 1_A \circ u = u,$$

that is, u is an idempotent. Since $\langle A, v\rangle$ is a generator of T, then in virtue of Proposition 10.3, $T \cong (h_A)_{\gamma^v}$, where γ^v is the closure operator associated with μ^v, that is, for every $b \in h_A B$, $\gamma^v(a) = (a *_T v)/_T v$. Furthermore, it is evident that $\mu^u : h_A \twoheadrightarrow h_A u$ is an epi, and hence

$h_A u \cong (h_A)_{\gamma^u}$, where γ^u is the closure operator associated with μ^u, that is, $\gamma^u(a) = (a \circ u)/_\circ u$. Then, it is enough to prove that $\gamma^v = \gamma^u$.

Note that $\gamma^v(\gamma^u(a)) = (((a \circ u)/u) * v)/v = (((a \circ u)/u) * (u \circ v))/v = ((((a \circ u)/u) \circ u) * v)/v = ((a \circ u) * v)/v = (a * (u * v))/v = (a * v)/v = \gamma^v(a)$, and then $\gamma^u \leqslant \gamma^v$. Moreover, $((a * v)/v) \circ u \leqslant a \circ u$, whence $\gamma^v(a) = (a * v)/v \leqslant (a \circ u)/u = \gamma^u(a)$, that is $\gamma^v \leqslant \gamma^u$.

In order to prove the other implication, observe that $\langle A, u\rangle$ is a generator of $h_A u$, since the morphism $\mu^u : h_A \twoheadrightarrow h_A u$ is an epi. And 2. of Definition 10.5 is satisfied, when replacing v by u. If moreover u is idempotent, then 1. is also satisfied, and hence $\langle A, u\rangle$ is a g-variable of $h_A u$. Finally, observe that having a g-variable is a property transferable by isomorphisms. $\square$

We will prove later that if a $\mathcal{Q}$-module has a g-variable, then all its generators are g-variables. Indeed, this will be part of the characterization of cyclic and projective $\mathcal{Q}$-modules as those having a g-variable (see Theorem 10.11). We split this result into the next two propositions for an easier exposition. But first, we prove the following lemma, which is the equivalent to Theorem 6.13 of [GT09].

Lemma 10.8. *If P has a g-variable $\langle A, v\rangle$ and u is the correspondent idempotent, then there exists a bijection between the $\mathcal{Q}$-morphisms $P \to T$ and the u-invariant elements of T, given by $\tau \mapsto \tau_A v$ and $y \mapsto \tau^y_B x = (x/_P v) *_T y$, for all $B \in \mathcal{Q}$, and all $x \in PB$.*

Proof. In one direction, if $\tau : P \to T$ is a $\mathcal{Q}$-morphism, then $u * \tau_A v = \tau_A(u * v) = \tau_A v$, that is, $\tau_A v$ is u-invariant.

On the other direction, if $y \in TA$ is an u-invariant, and we define $\tau^y_B x = (x/_P v) *_T y$, for all $B \in \mathcal{Q}$, and all $x \in PB$, then we will prove that $\tau^y : P \to T$ is a $\mathcal{Q}$-morphism. First, note that $\tau^y_B : PB \to TB$ is residuated, since for all $x \in PB$, and all $z \in TB$,

$$
\begin{aligned}
(x/v) * y \leqslant z \;&\Leftrightarrow\; x/v \leqslant z/y \\
&\Rightarrow\; x = (x/v) * v \leqslant (z/y) * v \\
&\Rightarrow\; x/v \leqslant \big((z/y) * v\big)/v \\
&\Rightarrow\; (x/v) \circ u \leqslant \big(((z/y) * v)/v\big) \circ u = (z/y) \circ u \\
&\Rightarrow\; (x/v) * y = (x/v) * (u * y) = \big((x/v) \circ u\big) * y \leqslant \big((z/y) \circ u\big) * y \\
&\qquad\qquad = (z/y) * (u * y) = (z/y) * y \leqslant y
\end{aligned}
$$

and moreover, its residuum is $(\tau^y_B)^+ : TB \to PB$ given by $(\tau^y_B)^+ z = (z/_T y) *_P v$. Now, in order to prove that $\tau^y : P \to T$ is a $\mathcal{Q}$-morphism, suppose that $a : B \to C$ is in $\mathcal{Q}$, and $x \in PB$. Then,

$$
\begin{aligned}
\tau^y_C(a * x) &= \big((a * x)/v\big) * y = \Big(\big((a \circ x/v) * v\big)/v\Big) * (u * y) \\
&= \Big(\big(((a \circ x/v) * v)/v\big) \circ u\Big) * y = \big((a \circ x/v) \circ u\big) * y \\
&= (a \circ x/v) * y = a * (x/v * y) = a * \tau^y_B x
\end{aligned}
$$

Finally, we see that this two correspondences are inverse to each other, since given $\tau : P \to T$, $\tau^{\tau_A v}_B x = (x/v) * \tau_A v = \tau_B(x/v * v) = \tau_B x$, and given an u-invariant y, we have $\tau^y_A v = (v/v) * y = \big(((1 * v)/v) \circ u\big) * y = (1 \circ u) * y = y$. $\square$

Proposition 10.9. *Let $\mathcal{Q}$ be a quantaloid and P a $\mathcal{Q}$-module. If P has a g-variable, then P is projective.*

Proof. Suppose that $\langle A, v\rangle$ is a g-variable of P, and $u : A \to A$ is the corresponding idempotent. Suppose that $\beta : T \twoheadrightarrow T'$ is an epi and that $\alpha : P \to T'$ is a $\mathcal{Q}$-morphism. Let us see that we can define a $\mathcal{Q}$-morphism $\overline{\alpha} : P \to T$ completing the diagram (6).

Since P has a g-variable, the $\mathcal{Q}$-morphism $\overline{\alpha} : P \to T$ that we are looking for should be determined by an u-invariant of T, $y = \overline{\alpha}_A v$. And by the commutativity of diagram (6), it should be satisfied that $\beta_A(\overline{\alpha}_A v) = \alpha_A v$. Then, we need to find an u-invariant $y \in TA$ such that $\beta_A y = \alpha_A v$. Since β is epi, in particular β_A is onto, and then there exists $t \in TA$ such that $\beta_A t = \alpha_A v$. Let $y = u * t$, which is evidently an u-invariant. Thus, $\beta_A y = \beta_A(u * t) = u * \beta_A t = u * \alpha_A v = \alpha_A(u * v) = \alpha_A v$. Therefore, y is the u-invariant we needed. Let $\overline{\alpha} = \tau^y$, with the notation of the previous lemma. We only have to prove that $\overline{\alpha}$ renders diagram (6) commutative. But, this is an easy calculation: for all $B \in \mathcal{Q}$, and $x \in PB$,

$$\beta_B \overline{\alpha}_B x = \beta_B \tau^y_B x = \beta_B\big((x/v) * y\big) = (x/v) * \beta_A y = (x/v) * \alpha_A v = \alpha_B\big((x/v) * v\big) = \alpha_B x.$$

□

Proposition 10.10. *Let $\mathcal{Q}$ be a quantaloid and P a cyclic and projective $\mathcal{Q}$-module. Then every generator of P is a g-variable.*

Proof. Let $\langle A, v\rangle$ be a generator of P, $\mu^v : h_A \twoheadrightarrow P$ the $\mathcal{Q}$-morphism given by the Yoneda Lemma, $\gamma = \widetilde{\mu^v}$ the closure operator on h_A associated with μ^v, and $\eta : (h_A)_\gamma \to P$ the isomorphism of Proposition 10.3. By the projectivity of P, the first diagram can be completed by an arrow $\alpha : P \to h_A$, and evaluating at A we obtain the second diagram,

$$\begin{tikzcd} & h_A \arrow[d, two heads, "\dot\gamma"] \\ P \arrow[ur, dashed, "\alpha"] \arrow[r, "\eta^{-1}"'] & (h_A)_\gamma \end{tikzcd} \qquad \begin{tikzcd} & {[A,A]} \arrow[d, two heads, "\dot\gamma_A"] \\ PA \arrow[ur, "\alpha_A"] \arrow[r, "\eta_A^{-1}"'] & {[A,A]_{\gamma_A}} \end{tikzcd} \qquad \begin{tikzcd} & u = \alpha_A(v) \arrow[d, mapsto] \\ v \arrow[ur, mapsto] \arrow[r, mapsto] & \gamma_A(1_A) \end{tikzcd}$$

In order to prove that $\langle A, v\rangle$ is indeed a g-variable, we see now that $u = \alpha_A(v)$ and v satisfy Conditions 1. and 2. of Definition 10.5. First, note that since $v = 1_A *_P v = \mu^v_A(1_A) = \eta_A\gamma_A(1_A)$, we obtain that $\gamma_A(1_A) = \eta_A^{-1}(v) = \gamma_A\alpha_A(v) = \gamma_A(u)$. Therefore, $u *_P v = \mu^v_A(u) = \eta_A\gamma_A(u) = \eta_A\gamma_A(1_A) = \mu^v_A(1_A) = v$. Furthermore, for every $a : A \to B$ in $\mathcal{Q}$, we have $\big((a *_P v)/_P v\big) \circ u = \gamma_B(a) *_{h_A} \alpha_A(v) = \alpha_B(\gamma_B(a) *_P v) = \alpha_B\mu^v_B\gamma_B(a) = \alpha_B\mu^v_B(\mu^v_B)^+\mu^v_B(a) = \alpha_B\mu^v_B(a) = \alpha_B(a *_P v) = a *_{h_A} \alpha_A(v) = a \circ u$. □

Theorem 10.11. *Let $\mathcal{Q}$ be a quantaloid and T a $\mathcal{Q}$-module. The following conditions are equivalent:*

(i) T is cyclic and projective.
(ii) T is cyclic and every generator is a g-variable.
(iii) T has a g-variable.
(iv) There exists an idempotent $u : A \to A$ in $\mathcal{Q}$ such that $T \cong h_A u$.

Proof. By Proposition 10.10, we have (i) $\Rightarrow$ (ii). The implication (ii) $\Rightarrow$ (iii) is evident. (iii) $\Leftrightarrow$ (iv) was proved in Proposition 10.7. And finally, (iii) $\Rightarrow$ (i) in virtue of Proposition 10.9. □

Corollary 10.12. *Let $\mathcal{Q}$ be a quantaloid. Then for every $A \in \mathcal{Q}$ the $\mathcal{Q}$-module h_A is cyclic and projective.*

Proof. It is immediate to see that, if $A \in \mathcal{Q}$, then $\langle A, 1_A\rangle$ is a generator of h_A, since $\mu^{1_A} : h_A \to h_A$ is the identity $\mathcal{Q}$-morphism in h_A, and therefore epi. Furthermore, taking $u = 1_A$, the equalities 1. and 2. of Definition 10.5 are trivially satisfied, showing that $\langle A, 1_A\rangle$ is a g-variable of h_A. □

Theorem 10.13. *Let $\mathcal{Q}$ be a small quantaloid. Then the category $\mathcal{Q}$-Mod has enough projectives, and dually enough injectives.*

Proof. We know that the category $\mathcal{Q}$-Mod has biproducts, and then the product of projectives is also the coproduct of projectives, and hence a projective. We will construct for every $\mathcal{Q}$-module T a projective H, which will be a product of projectives, and an epi $\varepsilon : H \twoheadrightarrow T$. Suppose that T is a $\mathcal{Q}$-module, and for every $A \in \mathcal{Q}$, let $H_A = h_A^{|TA|}$ be the power of h_A by the set $|TA|$ of the elements of TA, that is, the product $H_A = \prod_{x \in TA}(h_A)_x$, where $(h_A)_x = h_A$. For every $x \in TA$, let $\pi^x : H_A \to h_A$ the corresponding projection. Take now the product

$H = \prod_{A\in\mathcal{Q}} H_A$ and let $\pi^A : H \to H_A$ be the corresponding projection, for every $A \in \mathcal{Q}$. Therefore, for every $A \in \mathcal{Q}$, and every $x \in TA$ we have the $\mathcal{Q}$-morphism:

$$\varepsilon^{A,x} : H \xrightarrow{\pi^A} H_A \xrightarrow{\pi^x} h_A \xrightarrow{\mu^x} T.$$

Let $\varepsilon : H \to T$ be the join $\varepsilon = \bigvee\{\varepsilon^{A,x} : A \in \mathcal{Q}, x \in TA\}$ in $\mathrm{Hom}_{\mathcal{Q}}(H,T)$. It is easy to see that ε is an epi in $\mathcal{Q}$-Mod, since for every $B \in \mathcal{Q}$ and every $x_0 \in TB$, we can define the element $a \in HB$ determined in the following way: $a = (a_A : A \in \mathcal{Q})$, where for every $A \in \mathcal{Q}$, $a_A \in H_A B$, and $a_A = (a_{A,x} : x \in TA)$, where for every $x \in TA$, $a_{A,x} \in h_A B$, and

$$a_{A,x} = \begin{cases} \bot_{[A,B]} & \text{if } A \neq B, \\ \bot_{[B,B]} & \text{if } A = B \text{ and } x \neq x_0, \\ 1_B & \text{if } A = B \text{ and } x = x_0. \end{cases}$$

Therefore, it is straightforward to see that for every $A \in \mathcal{Q}$ and every $x \in TA$, $\varepsilon_B^{A,x}(a) = \mu_B^x \pi_B^x \pi_B^A(a) = a_{A,x} *_T x$, which has the value $\bot_{TB}$, unless $A = B$ and $x = x_0$, that takes the value x_0. Therefore, $\varepsilon_B(a) = \bigvee \varepsilon^{A,x}(a) = x_0$. Since every h_A is projective in virtue of Corollary 10.12, H is projective, and $\varepsilon : H \twoheadrightarrow T$ is the epi we were looking for. Thus, the category $\mathcal{Q}$-Mod has enough projectives, and by the duality property it also has enough injectives. $\square$

11 Equivalence of closure operators on $\mathcal{Q}$-Mod

Definition 11.1. Let $\mathcal{Q}$ be a quantaloid, T and T' two $\mathcal{Q}$-modules, and γ and δ closure operators on T and T', respectively. We say that γ and δ are *equivalent* if and only if there exists an isomorphism $\alpha : T_\gamma \to T'_\delta$. Such an isomorphism is called an *equivalence* between γ and δ.

Two $\mathcal{Q}$-morphism $\tau : T \to T'$ and $\rho : T' \to T$ are *mutually inverse* (with respect to γ and δ) if they satisfy $\dot{\delta} = \dot{\delta}\tau\rho$ and $\dot{\gamma} = \dot{\gamma}\rho\tau$. An equivalence $\alpha : T_\gamma \to T'_\delta$ is induced by τ and ρ if these are mutually inverse $\mathcal{Q}$-morphisms rendering commutative the following diagrams:

$$\begin{tikzcd} T \arrow[r, "\tau"] \arrow[d, two heads, "\dot{\gamma}"'] & T' \arrow[d, two heads, "\dot{\delta}"] \\ T_\gamma \arrow[r, "\alpha"] & T'_\delta \end{tikzcd} \qquad \begin{tikzcd} T & T' \arrow[l, "\rho"'] \arrow[d, two heads, "\dot{\delta}"] \\ T_\gamma \arrow[u, phantom, ""] \arrow[from=u, two heads, "\dot{\gamma}"'] & T'_\delta \arrow[l, "\alpha^{-1}"'] \end{tikzcd}$$

Theorem 11.2. *Let $\mathcal{Q}$ be a quantaloid, T and T' two $\mathcal{Q}$-modules, γ and ρ closure operators on T and T', respectively, and $\tau : T \to T'$ and $\rho : T' \to T$ two $\mathcal{Q}$-morphisms. Then the following statements are equivalent:*

(i) τ *and* ρ *induce an equivalence between* γ *and* δ*;*
(ii) $\gamma = \delta^\tau$ *and* $\dot{\delta} = \dot{\delta}\tau\rho$*;*
(iii) $\delta = \gamma^\rho$ *and* $\dot{\gamma} = \dot{\gamma}\rho\tau$*.*

Proof. Let us prove first (ii) $\Leftrightarrow$ (iii), and then we will prove (i) $\Leftrightarrow$ (ii). Suppose that $\gamma = \delta^\tau$ and $\dot{\delta} = \dot{\delta}\tau\rho$. Then, taking duals, we have $\rho^\partial\tau^\partial j = \rho^\partial\tau^\partial\dot{\delta}^\partial = (\dot{\delta}\tau\rho)^\partial = \dot{\delta}^\partial = j : (T'_\delta)^\partial \hookrightarrow T'^\partial$, the inclusion $\mathcal{Q}^{\mathrm{op}}$-morphism. In particular, for every $A \in \mathcal{Q}$, $\rho_A^+\tau_A^+ j_A = j_A : T'_\delta A \hookrightarrow T'A$, and therefore, $\delta_A = j_A\dot{\delta}_A = (\rho_A^+\tau_A^+ j_A)\dot{\delta}_A = \rho_A^+\tau_A^+\delta_A$. Thus,

$$\gamma_A^\rho = \rho_A^+\gamma_A\rho_A = \rho_A^+\delta_A^\tau\rho_A = \rho_A^+\tau_A^+\delta_A\tau_A\rho_A = \delta_A\tau_A\rho_A = \delta_A.$$

Moreover, $\gamma\rho\tau = \delta^\tau\rho\tau = \tau^\partial\delta\tau\rho\tau = \tau^\partial\delta\tau = \delta^\tau = \gamma$, which proves (ii) $\Rightarrow$ (iii). The implication in the other direction is also true by symmetry.

Now, assume (i), that is, τ and ρ induce an equivalence between γ and δ. In particular, α is a representation of γ into δ induced by τ. Thus, τ is an interpretation of γ into δ, that is $\gamma = \delta^\tau$. Moreover, $\dot{\delta}\tau\rho = \alpha\dot{\gamma}\rho = \alpha\alpha^{-1}\dot{\delta} = \dot{\delta}$.

It only remains to prove (ii) $\Rightarrow$ (i). Suppose that $\gamma = \delta^\tau$ and $\dot{\delta} = \dot{\delta}\tau\rho$. Since we have proved that (ii) $\Rightarrow$ (iii), we also have that $\delta = \gamma^\rho$ and $\dot{\gamma} = \dot{\gamma}\rho\tau$. Then, we have that τ and ρ are interpretations, and the induced representations are $\alpha = \dot{\delta}\tau i$ and $\alpha' = \dot{\gamma}\rho j$. We have then the equalities:

$$i\alpha'\alpha = i(\dot{\gamma}\rho j)(\dot{\delta}\tau i) = \gamma\rho\delta\tau i = \tau^+\delta\tau\rho\delta\tau i = \tau^+\delta\delta\tau i = \tau^+\delta\tau i = \gamma i = i,$$

and since i is injective, then $\alpha'\alpha = id_{T_\gamma}$. The other composition $\alpha\alpha' = id_{T'_\delta}$ follows by symmetry. Therefore, α is an equivalence that is induced by τ and its inverse is induced by ρ. $\square$

Theorem 11.3. *Let $\mathcal{Q}$ be a quantaloid, T and T' two projective $\mathcal{Q}$-modules, γ and ρ closure operators on T and T', respectively. Then every equivalence between γ and ρ is induced by mutually inverse interpretations.*

Proof. This is an immediate consequence of Theorem 9.10 and Proposition 9.9. $\square$

12 Modules induced by π-institutions

In this final section we explain how the theory of modules over quantaloids includes the theory of π-institutions. First we begin with actions in the sense of [GF06], that is, pairs $I = \langle \mathbf{Sign}, \mathrm{Sen}\rangle$, where $\mathrm{Sen} : \mathbf{Sign} \to \mathcal{S}et$ is a functor. A *translation* from $\langle \mathbf{Sign}_1, \mathrm{Sen}\rangle$ to $\langle \mathbf{Sign}_2, \mathrm{Sen}_2\rangle$ is a pair $\langle F, \alpha\rangle$ such that $F : \mathbf{Sign}_1 \to \mathbf{Sign}_2$ is a functor and $\alpha : \mathcal{P}\,\mathrm{Sen}_1 \to \mathcal{P}\,\mathrm{Sen}_2\,F$ is an additive natural map, that is, a natural map satisfying that for every $A \in \mathbf{Sign}_1$ and every set $X \subseteq \mathrm{Sen}_1\,A$, $\alpha_A X = \bigcup_{\varphi\in X} \alpha_A\{\varphi\}$. The category of actions is the category $\mathcal{A}ct$ with actions as objects and translations as morphisms. The composition of two translations $\langle F, \alpha\rangle$ and $\langle G, \beta\rangle$ is the translation $\langle GF, \beta_F \cdot \alpha\rangle$, where as usual β_F denotes the natural transformation obtained from β by precomposition with F, that is, $\beta_F : \mathcal{P}\,\mathrm{Sen}_2\,F \to \mathcal{P}\,\mathrm{Sen}_3\,GF$.

Given an action I, we will see that there exist a free quantaloid $\widehat{\mathcal{P}}\mathbf{Sign}$ generated over $\mathbf{Sign}$, and an enriched functor $\mathrm{Sen}^{\mathcal{P}} : \widehat{\mathcal{P}}\mathbf{Sign} \to \mathcal{S}\ell$, that is, a $\widehat{\mathcal{P}}\mathbf{Sign}$-module, $\mathrm{Sen}^{\mathcal{P}}$.

Note that for every quantaloid $\mathcal{Q}$, we can forget its enrichment and consider it just as a category, and in the same way we can forget the enrichment of morphisms of quantaloids and consider them just as functors. Hence, there exists a *forgetful* functor $U : \mathcal{S}\ell\text{-}\mathcal{C}at \to \mathcal{C}at$ forgetting the enrichment. It is well known that this forgetful functor has a left adjoint, that is, there exists a *free functor*.

Let us see how given a category $\mathcal{C}$, the *free quantaloid* $\widehat{\mathcal{P}}\mathcal{C}$ on $\mathcal{C}$ is obtained. The objects of $\widehat{\mathcal{P}}\mathcal{C}$ are the same as the objects of $\mathcal{C}$, and if A, B are objects of $\widehat{\mathcal{P}}\mathcal{C}$, then $\widehat{\mathcal{P}}\mathcal{C}(A, B) = \langle \mathcal{P}(\mathcal{C}(A, B)), \subseteq\rangle$, that is, an arrow from A to B in $\widehat{\mathcal{P}}\mathcal{C}$ is a set of arrows[4] from A to B in $\mathcal{C}$. Composition is defined as follows: if $a : A \to B$ and $b : B \to C$, then $b \circ a : A \to C$ is the set $b \circ a = \{g \circ f : f \in a, g \in b\}$. Is is easy to check that $\widehat{\mathcal{P}}\mathcal{C}$ is enriched over $\mathcal{S}\ell$. This quantaloid $\widehat{\mathcal{P}}\mathcal{C}$ is called the free quantaloid on $\mathcal{C}$ because it satisfies the following property:

Proposition 12.1. *For every locally-small category $\mathcal{C}$, there exists a functor $\{\}_{\mathcal{C}} : \mathcal{C} \to U(\widehat{\mathcal{P}}\mathcal{C})$ such that, for every quantaloid $\mathcal{Q}$ and every functor $F : \mathcal{C} \to U(\mathcal{Q})$, there exists a unique morphism of quantaloids $\overline{F} : \widehat{\mathcal{P}}\mathcal{C} \to \mathcal{Q}$ completing the diagram:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\{\}_{\mathcal{C}}"] \arrow[dr, "F"'] & U(\widehat{\mathcal{P}}\mathcal{C}) \arrow[d, dashed, "U(\overline{F})"] \\ & U(\mathcal{Q}) \end{tikzcd} \qquad \begin{tikzcd} \widehat{\mathcal{P}}\mathcal{C} \arrow[d, dashed, "\exists!\,\overline{F}"] \\ \mathcal{Q} \end{tikzcd}$$

[4]Note that, if this definition was taken literally, then the empty set would be an arrow shared by all the hom-sets. In order to avoid this and make hom-sets disjoint, we suppose as usual that the arrows are labeled with source and target.

Proof. The functor $\{\}_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{P}}\mathcal{C}$ is defined as the identity on objects, and for every $f : A \to B$ in $\mathcal{C}$, $\{\}_{\mathcal{C}}(f) = \{f\} : A \to B$ in $\widehat{\mathcal{P}}\mathcal{C}$.

If $F : \mathcal{C} \to U(\mathcal{Q})$ is a functor, we define $\overline{F} : \widehat{\mathcal{P}}\mathcal{C} \to \mathcal{Q}$ as F on objects, and for every $a : A \to B$ in $\widehat{\mathcal{P}}\mathcal{C}$, $\overline{F}(a) = \bigvee\{F(f) : f \in a\}$, where the supremum is taken in $[FA, FB]$. Since composition in $\mathcal{Q}$ is biresiduated, it is easy to see that $\overline{F}$ is a functor. Let us see that for every $A, B \in \widehat{\mathcal{P}}\mathcal{C}$, the restriction of $\overline{F}$ is a residuated map $\widehat{\mathcal{P}}\mathcal{C}(A, B) \to [FA, FB]$. Suppose that $a : A \to B$ is in $\widehat{\mathcal{P}}\mathcal{C}$, and $g : FA \to FB$ is in $\mathcal{Q}$. Then,

$$\begin{aligned} \overline{F}(a) \leqslant g \;&\Leftrightarrow\; \bigvee\{F(f) : f \in a\} \leqslant g \;\Leftrightarrow\; \forall f \in a,\ F(f) \leqslant g \\ &\Leftrightarrow\; a \subseteq \{f \in \mathcal{C}(A, B) : F(f) \leqslant g\}. \end{aligned}$$

Thus, we have that the map $\overline{F}^{+} : [FA, FB] \to \widehat{\mathcal{P}}\mathcal{C}(A, B)$ determined by the equality $\overline{F}^{+}(g) = \{f \in \mathcal{C}(A, B) : F(f) \leqslant g\}$ is the residuum of $\overline{F} : \widehat{\mathcal{P}}\mathcal{C}(A, B) \to [A, B]$. Finally, note that $\overline{F}$ is the unique morphism of quantaloids $\widehat{\mathcal{P}}\mathcal{C} \to \mathcal{Q}$ rendering the triangle commutative. $\square$

As usual, the existence of free objects ensures the existence of a left adjoint of the forgetful functor, that is, a *free functor*. It is defined in the following way: if $F : \mathcal{C} \to \mathcal{D}$ is a functor between two categories, then the following diagram can be completed, in virtue of the freeness of $\widehat{\mathcal{P}}\mathcal{C}$:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\{\}_{\mathcal{C}}"] \arrow[d, "F"'] \arrow[dr, "\{\}_{\mathcal{D}} \circ F"] & U(\widehat{\mathcal{P}}\mathcal{C}) \arrow[d, dashed, "U(\widehat{\mathcal{P}}F)"] & \widehat{\mathcal{P}}\mathcal{C} \arrow[d, dashed, "\widehat{\mathcal{P}}F"] \\ \mathcal{D} \arrow[r, "\{\}_{\mathcal{D}}"'] & U(\widehat{\mathcal{P}}\mathcal{D}) & \widehat{\mathcal{P}}\mathcal{D} \end{tikzcd} \tag{7}$$

Thus, if $a : A \to B$ is $\widehat{\mathcal{P}}\mathcal{C}$, then the image of a by $\widehat{\mathcal{P}}F$ can be easily calculated:

$$\begin{aligned} \widehat{\mathcal{P}}F(a) &= \overline{\{\}_{\mathcal{D}} \circ F}(a) = \bigvee\big\{\big(\{\}_{\mathcal{D}} \circ F\big)(f) : f \in a\big\} = \bigcup\big\{\{F(f)\} : f \in a\big\} \\ &= \{F(f) : f \in a\}. \end{aligned}$$

We have that, $\widehat{\mathcal{P}} : \mathcal{C}at \to \mathcal{S}\ell\text{-}\mathcal{C}at$ is a left adjoint of $U : \mathcal{S}\ell\text{-}\mathcal{C}at \to \mathcal{C}at$, and $\{\} : Id_{\mathcal{C}at} \to U\widehat{\mathcal{P}}$ is the unit of the adjunction $\widehat{\mathcal{P}} \dashv U$. The counit $\zeta : \widehat{\mathcal{P}}U \to Id_{\mathcal{S}\ell\text{-}\mathcal{C}at}$ can be readily obtained, taking into account that every component $\zeta_{\mathcal{Q}} : \widehat{\mathcal{P}}(U\mathcal{Q}) \to \mathcal{Q}$ is a morphism of quantaloids which is the identity on objects and respects arbitrary joins in hom-sets. Therefore, for every $a : A \to B$ in $\widehat{\mathcal{P}}(U\mathcal{Q})$, we have that $a \subseteq \mathcal{Q}(A, B)$, and $\zeta_{\mathcal{Q}}(a) = \bigvee a \in \mathcal{Q}(A, B)$.

The following construction is of special interest for us, since it allows encompassing π-institutions in the frame of modules over quantaloids, as we will see. It is well known that the functor $\mathcal{P} : \mathcal{S}et \to \mathcal{S}\ell$ is the free functor[5] of $\mathcal{S}\ell$, left adjoint of the forgetful functor $V : \mathcal{S}\ell \to \mathcal{S}et$, and the singleton maps $\sigma_A : A \to \mathcal{P}A$ are the insertions of generators, that is, the components of the unit of the adjunction. Using this functor we can construct, for every functor $F : \mathcal{C} \to \mathcal{S}et$ a new functor $F^{\mathcal{P}} : \widehat{\mathcal{P}}\mathcal{C} \to \mathcal{S}\ell$ as the unique morphism of quantaloids $\widehat{\mathcal{P}}\mathcal{C} \to \mathcal{S}\ell$ given by the freeness of $\widehat{\mathcal{P}}\mathcal{C}$ and rendering commutative the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\{\}_{\mathcal{C}}"] \arrow[d, "F"'] \arrow[dr, "\mathcal{P} \circ F"] & U(\widehat{\mathcal{P}}\mathcal{C}) \arrow[d, dashed, "U(F^{\mathcal{P}})"] & \widehat{\mathcal{P}}\mathcal{C} \arrow[d, dashed, "F^{\mathcal{P}}"] \\ \mathcal{S}et \arrow[r, "\mathcal{P}"'] & U(\mathcal{S}\ell) & \mathcal{S}\ell \end{tikzcd}$$

This functor acts in the following way: for every object A in $\mathcal{C}$, the image of A is the complete lattice $F^{\mathcal{P}}(A) = \mathcal{P}(FA)$, and for every arrow $a : A \to B$ in $\widehat{\mathcal{P}}\mathcal{C}$, $F^{\mathcal{P}}(a) : \mathcal{P}(FA) \to \mathcal{P}(FB)$ is determined by $x \mapsto \{F(f)\varphi : f \in a, \varphi \in x\}$. Note that if for one-object categories, this

[5]Here we do not consider $\mathcal{S}\ell$ as an enriched category, that is, $\mathcal{S}\ell$ denotes in fact $U(\mathcal{S}\ell)$.

construction amounts to the free quantale $\mathcal{P}(M)$ over a monoid M, and to the the usual lifting of M-set s to $\mathcal{P}(M)$-modules.

We also want to define for every translation a morphism relating the induced modules. Given two actions $I_1 = \langle \mathbf{Sign}_1, \mathrm{Sen}_1\rangle$ and $I_2 = \langle \mathbf{Sign}_2, \mathrm{Sen}_2\rangle$ with $\mathbf{Sign}_1 = \mathbf{Sign}_2$, it is easy to see that the additive natural transformations $\mathcal{P}\,\mathrm{Sen}_1 \to \mathcal{P}\,\mathrm{Sen}_2$ are exactly the $\widehat{\mathcal{P}\mathbf{Sign}_1}$-morphisms $\mathrm{Sen}_1^{\mathcal{P}} \to \mathrm{Sen}_2^{\mathcal{P}}$. They are the families $\tau = \{\tau_A : A \in \mathbf{Sign}\}$ such that $\tau_A : \mathcal{P}\,\mathrm{Sen}_1\,A \to \mathcal{P}\,\mathrm{Sen}_2\,A$ is a map respecting arbitrary unions, and rendering commutative the diagram

$$\begin{tikzcd} \mathcal{P}\,\mathrm{Sen}_1\,A \arrow[r, "\tau_A"] \arrow[d, "\mathcal{P}\,\mathrm{Sen}_1(f)"'] & \mathcal{P}\,\mathrm{Sen}_2\,A \arrow[d, "\mathcal{P}\,\mathrm{Sen}_2(f)"] \\ \mathcal{P}\,\mathrm{Sen}_1\,B \arrow[r, "\tau_B"] & \mathcal{P}\,\mathrm{Sen}_2\,B \end{tikzcd}$$

for every $f : A \to B$ in $\mathbf{Sign}_1$.

Nevertheless, if $\mathbf{Sign}_1 \neq \mathbf{Sign}_2$, then $\mathrm{Sen}_1^{\mathcal{P}}$ and $\mathrm{Sen}_2^{\mathcal{P}}$ are modules over different quantaloids, in general. Therefore, it is not possible to define a morphism of modules $\mathrm{Sen}_1^{\mathcal{P}} \to \mathrm{Sen}_2^{\mathcal{P}}$. We can surpass this obstacle, but we need to analyze a bit more the category of actions.

Observe that the category $\mathcal{A}ct$ is in fact the Grothendieck construction of a contravariant functor $\mathit{Fnct} : \mathcal{C}at \to \mathcal{C}at$ that we are going to define in the next paragraph. Therefore, it is a fibred category, and the first-projection functor $\mathfrak{p} : \mathcal{A}ct \to \mathcal{C}at$ is a split fibration. See Chapter 12 of [BW95] for the details about Grothendieck construtions and split fibrations and cofibrations.

The contravariant functor $\mathit{Fnct} : \mathcal{C}at \to \mathcal{C}at$ is defined in the following way: for every category $\mathbf{Sign}$, $\mathit{Fnct}(\mathbf{Sign})$ is the category that has as objects the $\mathcal{S}et$-valued functors $\mathrm{Sen} : \mathbf{Sign} \to \mathcal{S}et$, and an arrow $\alpha : \mathrm{Sen}_1 \to \mathrm{Sen}_2$ is an additive natural transformation $\alpha : \mathcal{P}\,\mathrm{Sen}_1 \to \mathcal{P}\,\mathrm{Sen}_2$. Given a functor $F : \mathbf{Sign}_1 \to \mathbf{Sign}_2$, a functor $\mathit{Fnct}(\mathbf{Sign}_2) \to \mathit{Fnct}(\mathbf{Sign}_1)$ is defined by sending Sen to $\mathrm{Sen}\,F$, and $\alpha : \mathrm{Sen}_1 \to \mathrm{Sen}_2$ to $\alpha_F : \mathrm{Sen}_1\,F \to \mathrm{Sen}_2\,F$. It follows immediately from the definitions that $\mathcal{A}ct = \mathit{Fnct}^{\sharp}$. We will see that from the perspective of modules over quantaloids, this functor has a more natural appearance.

It is easy to see that modules over quantaloids can also be gathered in a fibration, in the same way as modules over rings. This will allow communication between modules over different quantaloids.

Definition 12.2. Given two quantaloids $\mathcal{Q}$ and $\mathcal{Q}'$, and a morphism $F : \mathcal{Q} \to \mathcal{Q}'$, there is a standard procedure called *restriction of scalars* of transforming $\mathcal{Q}'$-modules into $\mathcal{Q}$-modules by composing with F:

$$\begin{tikzcd} \mathcal{Q} \arrow[rr, "F"] \arrow[dr, "T\circ F"'] & & \mathcal{Q}' \arrow[dl, "T"] \\ & \mathcal{S}\ell & \end{tikzcd}$$

In other words, for $a : A \to B$ and $x \in TFA$, $a *_{TF} x = F(a) *_T x$.

In fact, this is a functorial procedure, in the sense that there exists a contravariant functor $\_\text{-}\mathit{Mod} : \mathcal{S}\ell\text{-}\mathcal{C}at \to \mathcal{S}\ell\text{-}\mathcal{C}at$ determined by

$$\begin{tikzcd} \mathcal{Q} \arrow[r, mapsto] \arrow[d, "F"'] & \mathcal{Q}\text{-}\mathit{Mod} \\ \mathcal{Q}' \arrow[r, mapsto] & \mathcal{Q}'\text{-}\mathit{Mod} \arrow[u, "\_\circ F"'] \end{tikzcd}$$

This gives rise to a fibration $\mathfrak{q} : \mathit{Mod}^{\sharp} \to \mathcal{S}\ell\text{-}\mathcal{C}at$, the first projection of $\mathit{Mod}^{\sharp}$, the Grothendieck construction of $\_\text{-}\mathit{Mod}$ as a contravariant functor, that is, the objects of $\mathit{Mod}^{\sharp}$ are pairs $\langle \mathcal{Q}, T\rangle$, where $\mathcal{Q}$ is a quantaloid and T is a $\mathcal{Q}$-module, and the arrows are pairs $\langle F, \alpha\rangle : \langle \mathcal{Q}, T\rangle \to \langle \mathcal{Q}', T'\rangle$, where $F : \mathcal{Q} \to \mathcal{Q}'$ is a morphism of quantaloids and $\alpha : T \to T' \circ F$ is a $\mathcal{Q}$-morphism. Composition of arrows is defined by $\langle G, \beta\rangle\langle F, \alpha\rangle = \langle GF, \beta_F \cdot \alpha\rangle$.

Of course, if $\mathcal{Q}$ is a quantaloid, then the fiber over $\mathcal{Q}$, which is the subcategory $Mod^\sharp_{\mathcal{Q}}$ of $Mod^\sharp$ with objects pairs $\langle \mathcal{Q}, T\rangle$ and arrows of the form $\langle Id_{\mathcal{Q}}, \tau\rangle : \langle \mathcal{Q}, T\rangle \to \langle \mathcal{Q}, T'\rangle$, is isomorphic to $\mathcal{Q}$-Mod.

Then, every action $I = \langle \mathbf{Sign}, \mathrm{Sen}\rangle$ induces an object $I^{\mathcal{P}} = \langle \widehat{\mathcal{P}}\mathbf{Sign}, \mathrm{Sen}^{\mathcal{P}}\rangle \in Mod^\sharp$. Let us see that every translation $\langle F, \tau\rangle : I_1 \to I_2$ induces an arrow $\langle \widehat{\mathcal{P}}F, \tau\rangle : I_1^{\mathcal{P}} \to I_2^{\mathcal{P}}$ in $Mod^\sharp$. Recall that, given a functor $F : \mathcal{C} \to \mathcal{C}'$, there exists a functor $\widehat{\mathcal{P}}F : \widehat{\mathcal{P}}\mathcal{C} \to \widehat{\mathcal{P}}\mathcal{C}'$ that is defined as the morphism of quantaloids making commutative Diagram (7).

Proposition 12.3. *Let $I_1 = \langle \mathbf{Sign}_1, \mathrm{Sen}_1\rangle$ and $I_2 = \langle \mathbf{Sign}_2, \mathrm{Sen}_2\rangle$ be two actions and $\langle F, \tau\rangle :$ $I_1 \to I_2$ a translation. Then $\langle \widehat{\mathcal{P}}F, \tau\rangle : I_1^{\mathcal{P}} \to I_2^{\mathcal{P}}$ is a morphism in the category $Mod^\sharp$.*

Proof. First note that $\widehat{\mathcal{P}}F : \widehat{\mathcal{P}}\mathbf{Sign}_1 \to \widehat{\mathcal{P}}\mathbf{Sign}_2$ is a morphism of quantaloids, $\mathrm{Sen}_1^{\mathcal{P}}$ is a $\widehat{\mathcal{P}}\mathbf{Sign}_1$-module, $\mathrm{Sen}_2^{\mathcal{P}}$ is a $\widehat{\mathcal{P}}\mathbf{Sign}_2$-module, and by restriction of scalars, $\mathrm{Sen}_2^{\mathcal{P}} \circ \widehat{\mathcal{P}}F$ is a $\widehat{\mathcal{P}}\mathbf{Sign}_1$-module. According with the definition of $\mathrm{Sen}_2^{\mathcal{P}}$ and $\widehat{\mathcal{P}}F$, and since $U : \mathcal{S}\ell\text{-}\mathcal{C}at \to \mathcal{C}at$ is a functor, the following diagram commutes for $\mathcal{C} = \mathbf{Sign}_1$ and $\mathcal{D} = \mathbf{Sign}_2$:

$$\begin{tikzcd}
\mathcal{C} \arrow[rrr, "\{\}_{\mathcal{C}}"] \arrow[dr, "F"] \arrow[dd, bend right, "\mathrm{Sen}_2 F"'] & & & U(\widehat{\mathcal{P}}\mathcal{C}) \arrow[dl, "U(\widehat{\mathcal{P}}F)"] \arrow[dd, bend left, "U\left(\mathrm{Sen}_2^{\mathcal{P}} \circ \widehat{\mathcal{P}}F\right)"] \\
& \mathcal{D} \arrow[r, "\{\}_{\mathcal{D}}"] \arrow[dl, "\mathrm{Sen}_2"'] & U(\widehat{\mathcal{P}}\mathcal{D}) \arrow[dr, "U\left(\mathrm{Sen}_2^{\mathcal{P}}\right)"] & \\
\mathcal{S}et \arrow[rrr, "\mathcal{P}"] & & & U(\mathcal{S}\ell)
\end{tikzcd}
\qquad
\begin{tikzcd}
\widehat{\mathcal{P}}\mathcal{C} \arrow[d, "\widehat{\mathcal{P}}F"'] \arrow[dd, bend left, "\mathrm{Sen}_2^{\mathcal{P}} \circ \widehat{\mathcal{P}}F"] \\
\widehat{\mathcal{P}}\mathcal{D} \arrow[d, "\mathrm{Sen}_2^{\mathcal{P}}"'] \\
\mathcal{S}\ell
\end{tikzcd}$$

Therefore, $(\mathrm{Sen}_2)^{\mathcal{P}} \circ \widehat{\mathcal{P}}F = (\mathrm{Sen}_2 F)^{\mathcal{P}}$.

Since $\langle F, \tau\rangle : I_1 \to I_2$ is a translation, $\tau : \mathcal{P}\,\mathrm{Sen}_1 \to \mathcal{P}\,\mathrm{Sen}_2 F$ is an additive natural transformation. That is to say, a $\widehat{\mathcal{P}}\mathbf{Sign}_1$-morphism $\tau : \mathrm{Sen}_1^{\mathcal{P}} \to (\mathrm{Sen}_2 F)^{\mathcal{P}}$, and this ends the proof. $\square$

Then, we can define a functor $\zeta : \mathcal{A}ct \to Mod^\sharp$ by $\zeta\langle \mathbf{Sign}, \mathrm{Sen}\rangle = \langle \widehat{\mathcal{P}}\mathbf{Sign}, \mathrm{Sen}^{\mathcal{P}}\rangle$, and $\zeta\big(\langle F, \alpha\rangle\big) = \langle \widehat{\mathcal{P}}F, \alpha\rangle$. Note that ζ trivially respects identities, and moreover $\zeta\big(\langle G, \beta\rangle\big)\zeta\big(\langle F, \alpha\rangle\big) = \langle \widehat{\mathcal{P}}G, \beta\rangle\langle \widehat{\mathcal{P}}F, \alpha\rangle = \langle \widehat{\mathcal{P}}G\widehat{\mathcal{P}}F, \beta_{\widehat{\mathcal{P}}F} \cdot \alpha\rangle = \langle \widehat{\mathcal{P}}(GF), \beta_F \cdot \alpha\rangle = \zeta\big(\langle GF, \beta_F \cdot \alpha\rangle\big) = \zeta\big(\langle G, \beta\rangle\langle F, \alpha\rangle\big)$. Thus, we can see $\mathcal{A}ct$ as a subcategory of $Mod^\sharp$. Furthermore, the functor ζ respects the fibred structure of $\mathcal{A}ct$:

Theorem 12.4. *The pair $\langle \zeta, \widehat{\mathcal{P}}\rangle : \mathfrak{p} \to \mathfrak{q}$ is a morphism of split cofibrations.*

Proof. It is obvious that the follwoing diagram commutes:

$$\begin{tikzcd}
\mathcal{A}ct \arrow[r, "\zeta"] \arrow[d, "\mathfrak{p}"'] & Mod^\sharp \arrow[d, "\mathfrak{q}"] \\
\mathcal{C}at \arrow[r, "\widehat{\mathcal{P}}"] & \mathcal{S}\ell\text{-}\mathcal{C}at
\end{tikzcd}$$

We only have to prove that $\langle \zeta, \widehat{\mathcal{P}}\rangle$ respects the splittings of $\mathfrak{p}$ and $\mathfrak{q}$. Since these are Grothendieck constructions, their splittings are

$$\begin{aligned}
\kappa(F, \langle \mathbf{Sign}, \mathrm{Sen}\rangle) &= \langle F, id_{\mathcal{P}\,\mathrm{Sen}\,F}\rangle, && \text{for every } F \text{ with codomain } \mathbf{Sign}\text{, and} \\
\kappa'(F, \langle \mathcal{Q}, T\rangle) &= \langle F, id_T\rangle, && \text{for every } F \text{ with codomain } \mathcal{Q}.
\end{aligned}$$

Then, we have $\zeta\big(\kappa(F, \langle \mathbf{Sign}, \mathrm{Sen}\rangle)\big) = \zeta\big(\langle F, id_{\mathcal{P}\,\mathrm{Sen}\,F}\rangle\big) = \langle \widehat{\mathcal{P}}F, id_{\mathcal{P}\,\mathrm{Sen}\,F}\rangle = \langle \widehat{\mathcal{P}}F, id_{(\mathrm{Sen}\,F)^{\mathcal{P}}}\rangle = \langle \widehat{\mathcal{P}}F, id_{\mathrm{Sen}^{\mathcal{P}} \circ \widehat{\mathcal{P}}F}\rangle = \kappa'\big(\widehat{\mathcal{P}}F, \langle \widehat{\mathcal{P}}\mathbf{Sign}, \mathrm{Sen}^{\mathcal{P}}\rangle\big) = \kappa'\big(\widehat{\mathcal{P}}F, \zeta(\langle \mathbf{Sign}, \mathrm{Sen}\rangle)\big)$. $\square$

Let us see what happens to π-institutions. Recall that a π-institution is a pair $\langle I, C\rangle$, where I is an action and C is a closure operator on I. That is, C is a family $C = \{C_A : A \in \mathbf{Sign}\}$ such that for every $A \in \mathbf{Sign}$, C_A is a closure operator on $\operatorname{Sen} A$, and furthermore C satisfies the structurality property

$$\operatorname{Sen}(f) C_\Sigma \leqslant C_{\Sigma'} \operatorname{Sen}(f), \quad \text{for every } f : \Sigma \to \Sigma' \text{ in } \mathbf{Sign}. \tag{Str'}$$

Closure operators on the set $\operatorname{Sen} A$ coincide with closure operators on the complete lattice $\mathcal{P} \operatorname{Sen} A = \operatorname{Sen}^{\mathcal{P}} A$. Moreover, it is easy to see that the structurality property (Str) for C to be a closure operator on $\operatorname{Sen}^{\mathcal{P}}$ agrees with the structurality property (Str') for C to be a closure operator on I. Thus, we obtain the following result.[6]

Lemma 12.5. *Let $I = \langle \mathbf{Sign}, \operatorname{Sen}\rangle$ be an action. Then closure operators on I coincide with closure operators on the $\widehat{\mathcal{P}}\mathbf{Sign}$-module $\operatorname{Sen}^{\mathcal{P}}$.* □

Thus, π-institutions can be viewed as closure operators on a special kind of modules over free quantaloids. Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \operatorname{Sen}, \gamma\rangle$ is a π-institution and let $\operatorname{Th} : \mathbf{Sign} \to \mathcal{S}\ell$ be its theory functor, i.e., the functor determined by

$$\operatorname{Th} : \quad \begin{array}{ccccc} A & \longmapsto & \mathbf{Cl}(\gamma_A) & \qquad & \Gamma \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \operatorname{Th}(f)} & & \downarrow \\ B & \longmapsto & \mathbf{Cl}(\gamma_B) & & \gamma_B(\mathcal{P}\operatorname{Sen}(f)\Gamma) \end{array}$$

As we have said, if γ is a closure operator on the $\widehat{\mathcal{P}}\mathbf{Sign}$-module $\operatorname{Sen}^{\mathcal{P}}$, then $(\operatorname{Sen}^{\mathcal{P}})_\gamma$ is also a $\widehat{\mathcal{P}}\mathbf{Sign}$-module. Note that we have the following:

$$\operatorname{Th}(f)\Gamma = \gamma_B(\mathcal{P}\operatorname{Sen}(f)\Gamma) = \{f\} *_{(\operatorname{Sen}^{\mathcal{P}})_\gamma} \Gamma.$$

Then, we can formulate the question: what is the relation between $(\operatorname{Sen}^{\mathcal{P}})_\gamma$ and Th? In order to give an answer to this question, note that $\operatorname{Th} : \mathbf{Sign} \to \mathcal{S}\ell$ is a functor, and then there exists an enriched functor $\overline{\operatorname{Th}} : \widehat{\mathcal{P}}\mathbf{Sign} \to \mathcal{S}\ell$ by the freeness of $\widehat{\mathcal{P}}\mathbf{Sign}$, which is the canonical extension of Th from $\mathbf{Sign}$ to $\widehat{\mathcal{P}}\mathbf{Sign}$. Thus, this is a $\widehat{\mathcal{P}}\mathbf{Sign}$-module. Indeed, we have

$$(\operatorname{Sen}^{\mathcal{P}})_\gamma = \overline{\operatorname{Th}}.$$

Recall that the category of theories of a π-institution is the Grothendieck construction of its theory functor, $\mathbf{Th}\,\mathcal{I} = \operatorname{Th}^\sharp$. We can also build the Grothendieck construction of the functor $\overline{\operatorname{Th}}$. Then, which is the relation between $\mathbf{Th}\,\mathcal{I}$ and $\overline{\operatorname{Th}}^\sharp$? We will answer this question, but first we prove the following proposition, which states that the Grothendieck constructions of modules over quantaloids are quantaloids.

Proposition 12.6. *If $\mathcal{Q}$ is a quantaloid and T is a $\mathcal{Q}$-module, then $T^\sharp$ is a quantaloid.*

Proof. The category $T^\sharp$ has as objects the pairs $\langle A, x\rangle$, where $A \in \mathcal{Q}$ and $x \in TA$, and the arrows are pairs $\langle a, i\rangle : \langle A, x\rangle \to \langle B, y\rangle$, where $a : A \to B$ is in $\mathcal{Q}$, and $T(a)x \leqslant y$, that is $a *_T x \leqslant y$, and i is the pair $\langle a *_T x, y\rangle$. The composition of two arrows

$$\langle A, x\rangle \xrightarrow{\langle a,i\rangle} \langle B, y\rangle \xrightarrow{\langle b,j\rangle} \langle C, z\rangle$$

is the pair $\langle ba, k\rangle$, where $k = \langle (ba) *_T x, z\rangle$.

First, we prove that given two objects $\langle A, x\rangle$, $\langle B, y\rangle$ the order in $[A, B]$ induces an order in the hom-set $T^\sharp(\langle A, x\rangle, \langle B, y\rangle)$ rendering this set a complete lattice. We define $\langle a, i\rangle \leqslant \langle b, j\rangle$ if and only if $a \leqslant b$ in $[A, B]$. It is a partial order, evidently. Suppose now that we are given

[6] In virtue of this result, we are allowed to denote closure operators on actions with Greek letters γ, δ, ..., as we do for closure operators on modules.

a family of arrows $\{\langle a_k, i_k\rangle : \langle A, x\rangle \to \langle B, y\rangle\}_{k\in K}$. Then, for every $k \in K$, $a_k *_T x \leqslant y$, and therefore $(\bigvee a_k) *_T x = \bigvee(a_k *_T x) \leqslant y$. Thus, there exists an arrow $\langle \bigvee a_k, j\rangle : \langle A, x\rangle \to \langle B, y\rangle$ and it is immediate to check that this is the supremum of the given family in the lattice $T^\sharp\big(\langle A, x\rangle, \langle B, y\rangle\big)$.

The biresiduation property of the composition in $T^\sharp$ easily follows from the facts that composition in $\mathcal{Q}$ is biresiduated and for two arbitrary fixed objects $\langle A, x\rangle$ and $\langle B, y\rangle$, the arrows $\langle A, x\rangle \to \langle B, y\rangle$ are determined by their first components. □

Now we can answer the question we left open, and more generally, we answer the question: what is the relation between $\overline{L}^\sharp$ and $L^\sharp$, where $L : \mathcal{C} \to \mathcal{S}\ell$ is a functor.

Proposition 12.7. *Let $L : \mathcal{C} \to \mathcal{S}\ell$ a functor, $\overline{L} : \widehat{\mathcal{P}}\mathcal{C} \to \mathcal{S}\ell$ its extension to $\widehat{\mathcal{P}}\mathcal{C}$, and $L^\sharp$ and $\overline{L}^\sharp$ their corresponding Grothendieck constructions. Then*

$$\overline{L}^\sharp = \widehat{\mathcal{P}}(L^\sharp).$$

Proof. First note that the objects of $L^\sharp$ and $\overline{L}^\sharp$ are the same: the pairs $\langle A, x\rangle$, where $A \in \mathcal{C}$ (i.e., $A \in \widehat{\mathcal{P}}\mathcal{C}$) and $x \in LA = \overline{L}A$. The arrows $\langle A, x\rangle \to \langle B, y\rangle$ in $L^\sharp$ are those $f : A \to B$ such that $L(f)x \leqslant y$. The arrows $\langle A, x\rangle \to \langle B, y\rangle$ in $\overline{L}^\sharp$ are those $a : A \to B$ such that $\overline{L}(a)x \leqslant y$. That is to say, those $a \subseteq \mathcal{C}(A, B)$ such that for all $f \in a$, $L(f)x \leqslant y$. And this is the same as arbitrary subsets of arrows in $L^\sharp\big(\langle A, x\rangle, \langle B, y\rangle\big)$, what proves the result. □

Corollary 12.8. *If $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{Sen}, \gamma\rangle$ is a π-institution and* Th *is its theory functor, then*

$$\widehat{\mathcal{P}}(\mathbf{Th}\,\mathcal{I}) = \widehat{\mathcal{P}}(\mathrm{Th}^\sharp) = \overline{\mathrm{Th}}^\sharp = (\mathrm{Sen}^{\mathcal{P}})^\sharp_\gamma. \qquad \square$$

We are going to study now the different notions of a (semi-)interpretation and a (semi-)representation. We assume familiarity with these concepts in the settin of π-institutions. The reader can recall the definitions from [GF06]. First note the following: If δ is a closure operator on an action $I = \langle \mathbf{Sign}', \mathrm{Sen}\rangle$, then δ is a closure operator on the $\widehat{\mathcal{P}}\mathbf{Sign}'$-module $\mathrm{Sen}^{\mathcal{P}}$, as mentioned above. And, if $F : \mathbf{Sign} \to \mathbf{Sign}'$ is a functor, then δ_F is a closure operator on the $\widehat{\mathcal{P}}\mathbf{Sign}$-module $(\mathrm{Sen}\,F)^{\mathcal{P}}$.

Proposition 12.9. *Given two π-institutions, $\mathcal{I}_2 = \langle I_1, \gamma\rangle$ and $\mathcal{I}_2 = \langle I_2, \delta\rangle$, a translation $\langle F, \tau\rangle : I_1 \to I_2$ is a (semi-)interpretation of $\mathcal{I}_1$ into $\mathcal{I}_2$ if and only if τ is a (semi-)interpretation of γ into δ_F as closure operators on the $\widehat{\mathcal{P}}\mathbf{Sign}_1$-modules $\mathrm{Sen}_1^{\mathcal{P}}$ and $(\mathrm{Sen}_2\,F)^{\mathcal{P}}$, respectively.*

Proof. Since $\langle F, \tau\rangle$ is a translation of I_1 into I_2, then $\tau : \mathrm{Sen}_1^{\mathcal{P}} \to (\mathrm{Sen}_2\,F)^{\mathcal{P}}$ is a $\widehat{\mathcal{P}}\mathbf{Sign}_1$-morphism. Recall that $\langle F, \tau\rangle$ is a semi-interpretation of $\mathcal{I}_1$ into $\mathcal{I}_2$ if and only if for every $A \in \mathbf{Sign}$, $\tau_A\gamma_A \leqslant \delta_{FA}\tau_A$, or what is the same, $\gamma_A \leqslant \delta^\tau_{FA}$. And analogously for interpretations. □

Therefore, if $\langle F, \tau\rangle$ is a semi-interpretation of $\mathcal{I}_1$ into $\mathcal{I}_2$, τ is a semi-interpretation of γ into δ_F, and hence it induces a morphism of modules $\alpha : (\mathrm{Sen}_1^{\mathcal{P}})_\gamma \to ((\mathrm{Sen}_2\,F)^{\mathcal{P}})_{\delta_F}$ making commutative the following diagram:

$$\begin{tikzcd}
\mathrm{Sen}_1^{\mathcal{P}} \arrow[r, "\tau"] \arrow[d, "\dot{\gamma}"'] & (\mathrm{Sen}_2\,F)^{\mathcal{P}} \arrow[d, "\dot{\delta_F}"] \\
(\mathrm{Sen}_1^{\mathcal{P}})_\gamma \arrow[r, "\alpha"] & \big((\mathrm{Sen}_2\,F)^{\mathcal{P}}\big)_{\delta_F}
\end{tikzcd}$$

That is, if Th_1 and Th_2 are the theory functors of $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively, then α is a morphism of modules: $\alpha : \overline{\mathrm{Th}_1} \to \overline{\mathrm{Th}_2\,F}$. And hence, $\langle \widehat{\mathcal{P}}F, \alpha\rangle : \langle \widehat{\mathcal{P}}\mathbf{Sign}_1, \overline{\mathrm{Th}_1}\,\rangle \to \langle \widehat{\mathcal{P}}\mathbf{Sign}_2, \overline{\mathrm{Th}_2}\,\rangle$ is an arrow in $\mathit{Mod}^\sharp$. Furthermore, this induces a semi-representation of $\widehat{\mathcal{P}}(\mathbf{Th}\,\mathcal{I}_1)$ into $\widehat{\mathcal{P}}(\mathbf{Th}\,\mathcal{I}_2)$. In the case that $\langle F, \tau\rangle$ is a interpretation, $\langle \widehat{\mathcal{P}}F, \alpha\rangle$ induces a representation. In order to prove that, we first establish the following correspondences.

Proposition 12.10. *Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two quantaloids, and T_i a $\mathcal{Q}_i$-module, for $i = 1, 2$. Then every arrow $\langle F, \alpha\rangle : \langle \mathcal{Q}_1, T_1\rangle \to \langle \mathcal{Q}_2, T_2\rangle$ in $Mod^\sharp$ induces a signature-respecting morphism of quantaloids $\widetilde{F} : T_1^\sharp \to T_2^\sharp$ such that $\langle \widetilde{F}, F\rangle$ is a semi-representation of $T_1^\sharp$ into $T_2^\sharp$. Reciprocally, every semi-representation $\langle F, F^\circ\rangle$ of $T_1^\sharp$ into $T_2^\sharp$ such that F is a morphism of quantaloids induces an arrow $\langle F^\circ, \alpha\rangle : \langle \mathcal{Q}_1, T_1\rangle \to \langle \mathcal{Q}_2, T_2\rangle$ in $Mod^\sharp$. And these correspondences are inverse to each other. These correspondences remain inverse to each other when we restrict them to arrows $\langle F, \alpha\rangle$ such that α is mono and representations, respectively.*

Proof. Suppose that $\langle F, \alpha\rangle : \langle \mathcal{Q}_1, T_1\rangle \to \langle \mathcal{Q}_2, T_2\rangle$ is in $Mod^\sharp$. Then, by definition $F : \mathcal{Q}_1 \to \mathcal{Q}_2$ is a morphism of quantaloids and $\alpha : T_1 \to T_2 F$ is a $\mathcal{Q}_1$-morphism. We define $\widetilde{F} : T_1^\sharp \to T_2^\sharp$ by

$$\begin{tikzcd}
\langle A, x\rangle \arrow[r, mapsto] \arrow[d, "\langle a,i\rangle"'] & \langle FA, \alpha_A x\rangle \arrow[d, "\langle F(a),j\rangle"] \\
\langle B, y\rangle \arrow[r, mapsto] & \langle FB, \alpha_B y\rangle
\end{tikzcd}$$

This is well defined given that $\langle a, i\rangle : \langle A, x\rangle \to \langle B, y\rangle$ is in $T_1^\sharp$, and hence $a *_{T_1} x \leqslant y$, and applying the naturality of α, we have $F(a) *_{T_2} \alpha_A x = T_2\big(F(a)\alpha_A x\big) = T_2F(a)(\alpha_A x) = a *_{T_2F} \alpha_A x = \alpha_B(a *_{T_1} x) \leqslant \alpha_B y$. It is easy to check that this defines a functor that is signature-respecting.

In order to see that $\widetilde{F}$ is a morphism of quantaloids, suppose that $\langle A, x\rangle, \langle B, y\rangle \in T_1^\sharp$, and $\{\langle a_j, s_j\rangle : j \in J\}$ is a family in $[\langle A, x\rangle, \langle B, y\rangle]$. If we denote by t_j, t, and k the witnesses of $F(a_j) *_{T_2} \alpha_A x \leqslant \alpha_B y$, $F(\bigvee a_j) *_{T_2} \alpha_A x \leqslant \alpha_B y$, and $(\bigvee a_j) *_{T_1} x \leqslant y$, respectively, then it is straightforward to see that

$$\widetilde{F}\big(\bigvee \langle a_j, s_j\rangle\big) = \widetilde{F}\langle \bigvee a_j, k\rangle = \langle F(\bigvee a_j), t\rangle = \langle \bigvee F(a_j), t\rangle = \bigvee \langle F(a_j), t_j\rangle = \bigvee \widetilde{F}\langle a_j, s_j\rangle,$$

which shows that the restriction of $\widetilde{F}$ to every hom-set is residuated, and therefore $\widetilde{F}$ is a morphism of quantaloids.

The property that $\langle \widetilde{F}, F\rangle$ is a join morphism of split cofibrations agrees with the fact that α is a morphism of modules. Indeed, recall from Theorem 5.6 and Definition 5.8 of [GF06] that this is equivalent to the following two facts:

(i) the equation $\widetilde{F}_B T_1(a)x = T_2F(a)\widetilde{F}_A x$ holds for all $a : A \to B$ in $\mathcal{Q}_1$ and $x \in T_1 A$,

(ii) the equation $\widetilde{F}_A(\bigvee x_\lambda) = \bigvee \widetilde{F}_A x_\lambda$ holds for every $A \in \mathcal{Q}_1$ and for every family $\{a_\lambda : \lambda \in \Lambda\} \subseteq T_1 A$,

where $\widetilde{F}_A x$ is the second component of $\widetilde{F}\langle A, x\rangle$. That is $\widetilde{F}_A x = \alpha_A x$. Thus, (i) is $\alpha_B\big(T_1(a)x\big) = T_2F(a)\alpha_A x$, i.e., $\alpha_B(a *_{T_1} x) = a *_{T_2F} \alpha_A x$, whereas (ii) is $\alpha_A(\bigvee x_\lambda) = \bigvee \alpha_A x_\lambda$.

For the reverse situation, suppose that $\langle F, F^\circ\rangle$ is a semi-representation of $T_1^\sharp$ into $T_2^\sharp$, and for every $A \in \mathcal{Q}_1$ let α_A denote F_A. Note that, for every $A, B \in \mathcal{Q}_1$ and every family $\{a_\lambda : \lambda \in \Lambda\} \subseteq [A, B]$, there are arrows $\langle a_\lambda, \perp_B\rangle : \langle A, \perp_A\rangle \to \langle B, \perp_B\rangle$ in $T_1^\sharp$, and hence

$$\begin{aligned}
\big\langle F^\circ(\bigvee a_\lambda), 1_{\perp_{FB}}\big\rangle &= \big\langle F^\circ(\bigvee a_\lambda), F_A(1_{\perp_A})\big\rangle = F\big\langle \bigvee a_\lambda, 1_{\perp_B}\big\rangle = F\bigvee \langle a_\lambda, 1_{\perp_B}\rangle \\
&= \bigvee F\big(\langle a_\lambda, 1_{\perp_B}\rangle\big) = \bigvee \langle F^\circ a_\lambda, F_A(1_{\perp_A})\rangle = \big\langle \bigvee F^\circ a_\lambda, 1_{\perp_{FB}}\big\rangle,
\end{aligned}$$

where we obtain that $F^\circ(\bigvee a_\lambda) = \bigvee F^\circ a_\lambda$, and hence $F^\circ : \mathcal{Q}_1 \to \mathcal{Q}_2$ is a morphism of quantaloids. Finally, by an analogous argument, since $\langle F, F^\circ\rangle$ is join morphism of split cofibrations, we obtain that $\alpha : T_1 \to T_2F^\circ$ is a $\mathcal{Q}_1$-morphism.

One can readily prove the final assertion. □

Corollary 12.11. *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ two π-institutions, and* Th_1 *and* Th_2 *their corresponding theory functors. Then every arrow $\langle \widehat{\mathcal{P}}F, \alpha\rangle : \langle \widehat{\mathcal{P}}\mathbf{Sign}_1, \overline{\mathrm{Th}_1}\rangle \to \langle \widehat{\mathcal{P}}\mathbf{Sign}_2, \overline{\mathrm{Th}_2}\rangle$ in $Mod^\sharp$ induces a semi-representation of $\widehat{\mathcal{P}}(\mathbf{Th}\,\mathcal{I}_1)$ into $\widehat{\mathcal{P}}(\mathbf{Th}\,\mathcal{I}_2)$. If moreover α is mono, then the induced semi-representation is a representation.* □

Thus, every (semi-)representation of a π-institution $\mathcal{I}_1$ into another $\mathcal{I}_2$ induces a (semi-)representation of $\widehat{\mathcal{P}}(\mathbf{Th}\,\mathcal{I}_1)$ into $\widehat{\mathcal{P}}(\mathbf{Th}\,\mathcal{I}_2)$. All these results show how the study of representations and interpretations (and hence equivalence) between π-institutions can be done inside the theory of modules over quantaloids.

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