Differential Equations and Population Growth

November 24, 2009

A *differential equation* is simply an equation involving a function and its derivatives. Examples include:

$$\frac{dy}{dx} = 2y \text{ and } \frac{dP}{dt} = -P(1000 - P).$$

A *solution* to a differential equation is a function that satisfies the equations. As an example; $y = 2e^{2x}$ is solution to $f'(x) = 2f(x)$, as if $y = 2e^{2x}$, $\frac{dy}{dx} = 4e^{2x} = 2y$.

1. Verify that $x - \frac{1}{x}$ is a solution to the differential equation $xy' + y = 2x$.
2. For what values of r does $y = e^{rx}$ satisfy the differential equation $2y'' + y' - y = 0$?

A differential equation of the form $\frac{dy}{dx} = f(y)g(x)$ is called a 'separable' differential equation. As an example, consider $\frac{dy}{dx} = xy$. To solve this write:

$$\begin{aligned}
\frac{dy}{dx} &= xy \\
\frac{dy}{y} &= xdx \\
\int \frac{dy}{y} &= \int xdx \\
\ln(y) &= \frac{1}{2}x^2 + C \\
y &= Ke^{\frac{1}{2}x^2}.
\end{aligned}$$

Here $K = e^C$. To determine the 'proper' constant, an *initial value* is often given. For instance, solving $\frac{dy}{dx} = xy$ under the condition that $y(0) = 2$, we see that $K = 2$ is the proper constant.

1. Find the solution of $\frac{dy}{dx} = \sqrt{xy}$ with the initial condition $y(1) = 2$.
2. Find the solution of $\frac{dL}{dt} = kL^2 \ln(t)$, $L(1) = -1$. Note k is a constant; your answer will be in terms of k.

Population (as a function of time) is often modeled by a differential of the equation $\frac{dP}{dt} = P(M - P)$. Note that this is a separable differential equation.

1. Suppose $\frac{dP}{dt} = P(M-P)$. For what values of $'P'$ is $\frac{dP}{dt}$ positive? Negative? Zero? Does that make sense?

2. What is the physical interpretation of M? What happens to the population if it is above M? Below M, but positive?

3. Solve the separable differential equation $\frac{dP}{dt} = P(1000 - P)$. (Note, you will need to use partial fractions.)

4. Find the solution to the initial value problem $P(0) = 10$. Graph your solution. At what value $'t'$ is $P = 500$? $P = 900$? $P = 1000$?

5. Does it make sense that there was no $'t'$ where $P = 1000$ in the last question.

6. Find the solution to the initial value problem $P(0) = 1000$. Does this make sense?

7. Find the solution to the initial value problem $P(0) = 0$. Does this make sense?

8. What happens if $P(0) < 0$. Does this make sense?