Homework 2 Additional Questions

September 8, 2009

1.* Does $\sum_{i=1}^{\infty} \frac{\sin(i)}{i}$ converge or diverge?

2.* Does $\sum_{i=1}^{\infty} \frac{|\sin(i)|}{i}$ converge or diverge? HINT (for both): Considering the integral problem, (eg. $\int_1^{\infty} \frac{\sin x}{x} dx$ is the way to go. You can explicitly compute such integrals by integration by parts, but if you are not familiar with this method you can actually solve both problems anyways (by being a little clever).

3.** Suppose p_i is the ith prime, so $p_1 = 2, p_3 = 3, p_4 = 4 \ldots$. Show that $\sum_{i=1}^{\infty} \frac{1}{p_i}$ diverges. NOTE: This is quite hard in general.

4. Suppose I tell you that $p_n \sim n \ln(n)$ - in particular, that $\frac{1}{2} n \ln(n) \leq p_n \leq 2n \ln(n)$ for $n > n_0$. Show that $\sum_{i=1}^{\infty} \frac{1}{p_i}$ diverges. HINT: Just show that $\sum_{i=1}^{\infty} \frac{1}{i \ln(i)}$ diverges. Why is this enough?

5.** Suppose I define a function $f(x)$ such that:

$$f(x) = \begin{cases} x & \text{if } x \leq e \\ x f(\ln x) & \text{if } x > e. \end{cases}$$

Does $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converge? (Give a proof!)

NOTE: This problem appeared on the 2008 Putnam exam, which is a (notoriously tricky) mathematics competition for undergraduates. If you enjoy solving problems like this, I highly recommend that you look into participating in this competition - awards go to high scorers, Emory's math department has a prize for the highest scorer at Emory. Note that the most common score on the exam is a 0, so it is quite tough! Extra credit is available for solving this problem (of course, since the solution is online, I trust that you will still work it out for yourself!)

6.***** An arithmetic progression is a sequence of terms with a common difference. Examples are 2, 4, 6, 8 (common difference is 2), or 10, 17, 24 (common difference 7.) Prove or disprove the following conjecture of Erdős and Rényi: Suppose S is a sequence of numbers $s_1, s_2, s_3, \ldots$. If $\sum_{i=1}^{\infty} \frac{1}{s_i}$ diverges then S contains arithmetic progressions of arbitrarily long length.

Examples: If S is the prime numbers this is true (due to Green and Tao). For instance, $3, 5, 7$ is an arithmetic progression of length 3 in the primes. If S is the powers of 2, then the sum converges and indeed there are no arithmetic progressions of any length > 2 in S.

NOTES: This is an *unsolved* problem, and hence very difficult! There is a $ 3000 prize for a solution to this problem, offered by Erdős. Terry Tao won the Fields medal (think Nobel Prize in math) largely for his work on showing that the primes contained arbitrarily long arithmetic progressions. A good starting place would be to show that (if $\sum_{i=1}^{\infty} \frac{1}{s_i}$ diverges) there are arithmetic progressions of length 3. This is a good, fun, but hard problem to think about!