Math 361, Problem Set 1 Solutions

September 10, 2010

1. (1.2.9) If $C_1, C_2, C_3, \ldots$ are sets such that $C_k \supseteq C_{k+1}$, $k = 1, 2, 3, \ldots, \infty$, we define $\lim_{k\to\infty} C_k$ as the intersection $\bigcap_{k=1}^{\infty} C_k = C_1 \cap C_2 \cap \ldots$. Find $\lim_{k\to\infty} C_k$ for the following, and draw a picture of a typical 'C_k' on the line or plane, as appropriate:

   a. $C_k = \{x : 2 - 1/k < x \leq 2\}$, $k = 1, 2, 3, \ldots$.

   b. $C_k = \{x : 2 < x \leq 2 + \frac{1}{k}\}$, $k = 1, 2, 3, \ldots$.

   c. $C_k = \{(x, y) : 0 \leq x^2 + y^2 \leq \frac{1}{k}\}$, $k = 1, 2, 3, \ldots$.

   Note: In addition to the book problem, I ask for a picture of the set.

   Answer: For (a), note that $2 \in C_k$ for every k, but $2 - \epsilon$ is not in every C_k for any $\epsilon > 0$. This is because if $k > \frac{1}{\epsilon}$, $2 - \epsilon \notin C_k$. Therefore $\lim_{k\to\infty} C_k = \bigcap_{k=1}^{\infty} C_k = \{2\}$.

   For (b) note that $2 \notin C_k$ for any k. As before, $2 + \epsilon$ is not in C_k for $k > 1/\epsilon$. Therefore $\lim_{k\to\infty} C_k = \emptyset$.

2. (1.2.4) Let Ω denote the set of points interior to or on the boundary of a cube with edge of length 1. Moreover, say the cube is in the first octant with one vertex at the point $(0, 0, 0)$ and an opposite vertex at the point $(1, 1, 1)$. Let $Q(C) = \int\int\int_C dxdydz$.

   (a.) If $C \subseteq \Omega$ is the set $\{(x, y, z) : 0 < x < y < z < 1\}$ compute $Q(C)$. Describe the set C in words (or picture).

   (b.) If $C \subseteq \Omega$ is the set $\{(x, y, z) : 0 < x = y = z < 1\}$ compute $Q(C)$. Describe the set C in words (or picture).

   Answer: For (a) note that C is a triangular prism and :

$$\begin{aligned} Q(C) = \int_{z=0}^{1} \int_{y=0}^{z} \int_{x=0}^{y} dxdydz &= \int_{z=0}^{1} \int_{y=0}^{z} ydydz \\ &= \int_{z=0}^{1} \frac{z^2}{2} dz = \frac{1}{6}. \end{aligned}$$

For (b) note that C has dimension one, and thus is a line. Therefore $Q(C) = 0$ as a line has no volume. Alternately

$$Q(C) = \int_{z=0}^{1} \int_{y=z}^{z} \int_{x=z}^{z} dxdydz = 0.$$

3. (1.3.6) Suppose $\Omega = \mathbb{R}$. For $C \subseteq \Omega$ such that $\int_C e^{-|x|}dx$ exists, define $Q(C) = \int_C e^{-|x|}dx$. Show that $Q(C)$ is not a probability set function. What constant do we need to multiply the integrand by to make $Q(C)$ a probability set function?

   Answer: Note that

   $$\int_\Omega e^{-|x|}dx = \int_{-\infty}^{\infty} e^{-|x|}dx = 2\int_0^\infty e^{-x}dx = 2.$$

   Therfore we meed to multiply the integrand by $\frac{1}{2}$ to ensure that $Q(\Omega) = 1$ and that $Q(C)$ is a probability set function.

4. (1.3.10) Suppose we turn over cards simultaneously from two well shuffled decks of ordinary playing cards. We say we obtain an exact match on a particular turn if the same card appears from each deck; for example the queen of spades against the queen of spades. Let p_M equal the probability of at least one exact match.

   (a.) Show that
   $$p_M = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots - \frac{1}{52!}.$$
   Note: There is a hint in the book.

   (b.) Show that $p_M \approx 1 - e^{-1}$.

   (c.) * Suppose instead of requiring the exact same card for a match, we only require two cards to have the same rank (that is both kings). Let p'_M denote the probability of at least one match. What is p'_M?

   Let A_i denote the event that there is a match at the ith position. Then

   $$\mathbb{P}(A_i) = \frac{1}{52},$$

   as whatever card I draw from the first deck the card I draw from the second has a $1/52$ chance of matching it. For A_i, A_j with $i \neq j$

   $$\mathbb{P}(A_i \cap A_j) = \frac{1}{52 \cdot 51},$$

   Here the first card has $\frac{1}{52}$ chance of matching, and the second a $\frac{1}{51}$ (as there are 51 remaining cards.)

In general:

$$\mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k}) = \frac{1}{52 \cdot 51 \cdot \cdots \cdot (52 - k + 1)}$$

for the same reason.

We apply inclusion exclusion. Remember that

$$\begin{aligned} p_k = \sum_{i_1, \ldots, i_k} \mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k}) &= \binom{52}{k} \frac{1}{52 \cdot 51 \cdot \cdots \cdot (52 - k + 1)} \\ &= \frac{1}{k!}. \end{aligned}$$

Answer

(a) then is exactly the inclusion exclusion formula.

(b) follows from the Taylor series

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \ldots$$

evaluated at $x = 1$.

(c) was a mistake to ask, let us never speak of it again.

5. Recall that the *Borel σ-field* $\mathcal{B}$ is defined as follows: Let $\mathcal{C} = \{(a, b) : a, b \in \mathbb{R} \cup \{\infty\}\}$. Then $\mathcal{B} = \sigma(\mathcal{B})$, the smallest σ-field containing $\mathcal{C}$.

   Define $\mathcal{C}' = \{[a, b] : a, b \in \mathbb{R} \cup \{\infty\}\}$. Let $\mathcal{B}' = \sigma(\mathcal{C}')$. Show that $\mathcal{B} = \mathcal{B}'$. **Hint:** It suffices to show that $\mathcal{C} \subseteq \mathcal{B}'$ and $\mathcal{C}' \subseteq \mathcal{B}$. Why?

   Answer We want to show that the open intervals (a, b) are in $\mathcal{B}'$ and the closed intervals $[a, b]$ are in $\mathcal{B}$. Note that:

   $$(a, b)^c = (-\infty, a] \cup [b, \infty)$$

   Since $(a, b)^c$ is the union of two closed intervals (in $\mathcal{C}'$ and hence in $\mathcal{B}'$), this tells us that $(a, b)^c \in \mathcal{B}'$. Since $\mathcal{B}'$ is a σ-field, this tells us that $(a, b) = ((a, b)^c)^c \in \mathcal{B}'$ as σ-fields are closed under complements.

   Likewise:

   $$[a, b]^c = (-\infty, a) \cup (b, \infty)$$

   Thus $[a, b]^c$ is the union of two open intervals, and hence in $\mathcal{B}$. But then $[a, b] = ([a, b]^c)^c \in \mathcal{B}$, as desired.

   Since $\mathcal{C}' \subseteq \mathcal{B}$ and $\mathcal{B}'$ is the *smallest* σ-field containing $\mathcal{C}'$, this implies that $\mathcal{B}' \subseteq \mathcal{B}$. (That is, in the big intersection that defines $\mathcal{B}'$, $\mathcal{B}$ is one of the terms). For exactly the same reason, $\mathcal{B} \subseteq \mathcal{B}'$. But then $\mathcal{B} = \mathcal{B}'$.

6. (1.3.16) In a lot of 50 light bulbs, there are 2 bad bulbs. An inspector examines 5 bulbs, which are selected at random and without replacement.

(a.) Find the probability that at least one defective bulb being among the five.

(b.) Instead of five bulbs, the examiner chooses k bulbs at random and without replacement. How large must k be to ensure that the probability of finding at least one bad bulb exceeds $\frac{1}{2}$.

(c.) Instead of choosing k bulbs at random and without replacement, the inspector chooses k bulbs at random, *with* replacement. How large must k be to ensure that the probabily of finding at least one bad bulb exceeds $\frac{1}{2}$.

Answer:

For (a),

$$p = 1 - \frac{\binom{48}{5}}{\binom{50}{5}}$$

For (b) the question is when is

$$p = 1 - \frac{\binom{48}{k}}{\binom{50}{k}} > \frac{1}{2}$$

Equivalently, when is

$$\frac{1}{2} > \frac{\binom{48}{k}}{\binom{50}{k}} = \frac{(48-k+2)(48-k+1)}{50 * 49}.$$

This occurs first when $k = 15$.

For (c) we instead want

$$\left(\frac{48}{50}\right)^k < 1/2.$$

This occurs when $k = 17$ (though, in this case, only barely!).

Suggested problems: 1.2.3, 1.2.5, 1.2.6, 1.3.1, 1.3.9, 1.3.11-1.3.15 (not to be turned in).