Math 361, Problem Set 1

August 27, 2010

1. (1.2.9) If $C_1, C_2, C_3, \ldots$ are sets such that $C_k \supseteq C_{k+1}$, $k = 1, 2, 3, \ldots, \infty$, we define $\lim_{k\to\infty} C_k$ as the intersection $\bigcap_{k=1}^{\infty} C_k = C_1 \cap C_2 \cap \ldots$. Find $\lim_{k\to\infty} C_k$ for the following, and draw a picture of a typical 'C_k' on the line or plane, as appropriate:

   a. $C_k = \{x : 2 - 1/k < x \leq 2\}$, $k = 1, 2, 3, \ldots$.
   b. $C_k = \{x : 2 < x \leq 2 + \frac{1}{k}\}$, $k = 1, 2, 3, \ldots$.
   c. $C_k = \{(x, y) : 0 \leq x^2 + y^2 \leq \frac{1}{k}\}$, $k = 1, 2, 3, \ldots$.

   Note: In addition to the book problem, I ask for a picture of the set.

2. (1.2.4) Let Ω denote the set of points interior to or on the boundary of a cube with edge of length 1. Moreover, say the cube is in the first octant with one vertex at the point $(0, 0, 0)$ and an opposite vertex at the point $(1, 1, 1)$. Let $Q(C) = \int\int\int_C dxdydz$.

   (a.) If $C \subseteq \Omega$ is the set $\{(x, y, z) : 0 < x < y < z < 1\}$ compute $Q(C)$. Describe the set C in words (or picture).

   (b.) If $C \subseteq \Omega$ is the set $\{(x, y, z) : 0 < x = y = z < 1\}$ compute $Q(C)$. Describe the set C in words (or picture).

   Note: In addition to the book problem, I ask for a desciption of the set.

3. (1.3.6) Suppose $\Omega = \mathbb{R}$. For $C \subseteq \Omega$ such that $\int_C e^{-|x|}dx$ exists, define $Q(C) = \int_C e^{-|x|}dx$. Show that $Q(C)$ is not a probability set function. What constant do we need to multiply the integrand by to make $Q(C)$ a probability set function?

4. (1.3.10) Suppose we turn over cards simultaneously from two well shuffled decks of ordinary playing cards. We say we obtain an exact match on a particular turn if the same card appears from each deck; for example the queen of spades against the queen of spades. Let p_M equal the probability of at least one exact match.

   (a.) Show that
   $$p_M = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots - \frac{1}{52!}.$$
   Note: There is a hint in the book.

(b.) Show that $p_M \approx 1 - e^{-1}$.

(c.) * Suppose instead of requiring the exact same card for a match, we only require two cards to have the same rank (that is both kings). Let p'_M denote the probability of at least one match. What is p'_M?

5. Recall that the *Borel σ-field* $\mathcal{B}$ is defined as follows: Let $\mathcal{C} = \{(a, b) : a, b \in \mathbb{R} \cup \{\infty\}\}$. Then $\mathcal{B} = \sigma(\mathcal{B})$, the smallest σ-field containing $\mathcal{C}$.

   Define $\mathcal{C}' = \{[a, b] : a, b \in \mathbb{R} \cup \{\infty\}\}$. Let $\mathcal{B}' = \sigma(\mathcal{C}')$. Show that $\mathcal{B} = \mathcal{B}'$. **Hint:** It suffices to show that $\mathcal{C} \subseteq \mathcal{B}'$ and $\mathcal{C}' \subseteq \mathcal{B}$. Why?

6. (1.3.16) In a lot of 50 light bulbs, there are 2 bad bulbs. An inspector examines 5 bulbs, which are selected at random and without replacement.

   (a.) Find the probability that at least one defective bulb being among the five.

   (b.) Instead of five bulbs, the examiner chooses k bulbs at random and without replacement. How large must k be to ensure that the probability of finding at least one bad bulb exceeds $\frac{1}{2}$.

   (c.) Instead of choosing k bulbs at random and without replacement, the inspector chooses k bulbs at random, *with* replacement. How large must k be to ensure that the probabily of finding at least one bad bulb exceeds $\frac{1}{2}$.

Suggested problems: 1.2.3, 1.2.5, 1.2.6, 1.3.1, 1.3.9, 1.3.11-1.3.15 (not to be turned in).