Math 361, Problem set 11

Due 11/6/10

1. (3.4.32) Evaluate $\int_2^3 \exp(-2(x-3)^2)dx$ - without a calculator. Use the appendix table. *Answer:*

   Note that if X has a $N(3, \frac{1}{2})$ distribution then, X has pdf

   $$f_X(x) = \frac{2}{\sqrt{2\pi}} \exp(-2(x-3)^2), \qquad -\infty < x < \infty$$

   Thus

   $$\int_2^3 \exp(-2(x-3)^2)dx = \frac{\sqrt{2\pi}}{2}\mathbb{P}(2 < X < 3) = \frac{\sqrt{2\pi}}{2}\mathbb{P}(\frac{2-3}{1/2} < \frac{X-3}{2} < 0) = \frac{\sqrt{2\pi}}{2}\mathbb{P}(-2 < Z < 0) = \frac{\sqrt{2\pi}}{2}$$

2. (3.4.19) Let the random variable X have a distribution that is $N(\mu, \sigma^2)$.

   (a) Does the random variable $Y = X^2$ also have a normal distribution?

   (b) Would the random variable $Y = aX + b$ have a normal distribution?

   Answer: $Y = X^2$ does not; $Y \geq 0$ and a normal random variable is negative with positive probability.

   $Y = aX+b$ is $N(a\mu+b, a^2\sigma^2)$. We have that $Y = g(x)$ where $g(x) = ax+b$. Thus $g^{-1}(y) = \frac{y-b}{a}$, and $(g^{-1})'(y) = 1/a$. Thus since

   $$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

   we have that

   $$f_Y(y) = |f_X(g^{-1}(y))(g^{-1})'(y)| = \frac{1}{|a|\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-a\mu-b}{a\sigma}\right)^2\right).$$

   Note that we technically need the $|\cdot|$ in the case where $a < 0$.

3. (3.4.22) Let $f(x)$ and $F(x)$ be the pdf and the cdf of a distribution of the continuous type such that $f'(x)$ exists for all x. Let the mean of the truncated distribution that has pdf $g(y) = f(y)/F(b), -\infty < y < b$, zero elsewhere, be equal to $-f(b)/F(b)$ for all real b. Prove that $f(x)$ is a pdf of a standard normal distribution.

   Answer

   We have the relationship

   $$\int_{-\infty}^{b} y\frac{f(y)}{F(b)}dy = \frac{-f(b)}{F(b)}$$

   We cancel $F(b)$ from both sides to get

   $$\int_{-\infty}^{b} yf(y)dy = -f(b).$$

   Taking the derivative of both sides, applying the FTC we have

   $$bf(b) = -f'(b).$$

   This is a separable differential equation, $by = \frac{dy}{db}$. Solving we have:

   $$\begin{aligned} \int -bdb &= \int \frac{dy}{y} \\ \frac{-b^2}{2} + C &= \ln y \\ f(b) = y &= \exp(-b^2/2 + C) = K\exp(-b^2/2). \end{aligned}$$

   But $f(b)$ is a pdf, and hence the only constant that K can be is the one that makes this integrate to 1: $K = \frac{1}{\sqrt{2\pi}}$. Therefore $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, the pdf of a standard normal distribution.

4. (4.2.4) Let $X_1, \ldots, X_n$ be iid random varaibles with common pdf with $f(x) = e^{-x-\theta}$ for $x > \theta$, 0 elsewhere, where $-\infty < \theta < \infty$ is fixed. This pdf is called the shifted exponential. Let $Y_n = \min\{X_1, \ldots, X_n\}$. Prove that $Y_n \to \theta$ in probability, by obtaining the cdf and pdf of Y_n. (Note: if you can do it without obtaining the pdf and cdf of Y_n, that's fine too.)

   Answer:

   Note that $\mathbb{P}(X_i > \theta + \epsilon) = \int_{\theta+\epsilon}^{\infty} e^{-(x-\theta)} = e^{-\epsilon} < 1$ if $\epsilon < 1$. If $|Y_n - \theta| > \epsilon$ then it must be that $X_i > \theta + \epsilon$ for every single $i = 1 \ldots n$. Since the X_i are independent,

   $$\mathbb{P}(|Y_n - \theta| > \epsilon) = \prod_i = 1^n \mathbb{P}(X_i > \theta + \epsilon) = e^{-\epsilon n} \to 0$$

   as $n \to \infty$. Thus $Y_n \to_p \theta$.

5. (4.3.11) Let the random variable Z_n have a Poisson distribution with parameter $\mu = n$. Show that the limiting distribution of the random variable $Y_n = (Z_n - n)/\sqrt{n}$ is normal with mean zero and variance 1.

   Answer $M_{Z_n} = e^{n(e^t-1)}$. We have that the moment generating function $M_n(t)$ of $\frac{Z_n-n}{\sqrt{n}}$ is

   $$M_n(t) = \mathbb{E}[e^{t\frac{Z_n-n}{\sqrt{n}}}] = e^{-t\sqrt{n}}M(t/\sqrt{n}) = e^{-t\sqrt{n}}e^{n(e^{t/\sqrt{n}}-1)}.$$

   Expanding the Taylor series we have that

   $$\begin{aligned} M_n(t) &= e^{-t\sqrt{n}}e^{n(1+\frac{t}{\sqrt{n}}+\frac{t^2}{2n}+\frac{t^3}{3!n^{3/2}}+\cdots-1)} \\ &= e^{-t\sqrt{n}}e^{t\sqrt{n}+\frac{t^2}{2}+\frac{t^3}{3!\sqrt{n}}+\cdots} \\ &= e^{t^2/2+\frac{t^3}{3!\sqrt{n}}+\cdots} \to e^{t^2/2} \end{aligned}$$

   as $n \to \infty$. But this is the MGF of the standard normal so we are done.

6. (4.3.7) Let X_n have a gamma distribution with parameter $\alpha = n$ and β where β is not a function of n. Let $Y_n = X_n/n$. Find the limiting distribution of Y_n. Hint: Find the mgf of Y_n (not bad as we found the mgf of X_n in class). Take the limit and figure out what kind of distribution leads to this new mgf. Hint 2: this is the MGF of a *constant* function. What constant? Why does this make sense?

   Answer: I am not so concerned as whether you consider Y_n to be the sum of n exponential β's or n exponential $1/\beta$. The book actually treats this gamma as the sum of exponential $\frac{1}{\beta}$'s, and it's not so important to the method. I will use the standard I did in class where $\gamma(n, \beta)$ is the sum of n exponential(β)'s. (But the answer will effectively be the same).

   $$M_{Y_n}(t) = \left(\frac{\beta}{\beta - t/n}\right)^n = \left(\frac{\beta n}{\beta n - t}\right)^n = \left(1 + \frac{t}{\beta n - t}\right)^n \to e^{\frac{t}{\beta}}$$

   This is the MGF of the constant $1/\beta$. This makes sense, as by the weak law of large numbers the average of exponential(β) random variables tends to $1/\beta$.

   (Note that if you used the exact same MGF in the book you would have gotten that Y_n tended in distribution to the constant β. There is an annoying inconsistency as in both the Γ and exponential distribution where sometimes the parameter is the mean, and sometimes it is one over the mean.).

7. (4.4.6) Let Y be $Bin(400, \frac{1}{5})$. Compute an approximate value of $\mathbb{P}(0.25 < Y/400)$.

   Note $\mathbb{E}[Y] = 80$ and $\mathrm{Var}(Y) = np(1-p) = 64$, so $\sigma = 8$. Thus $\frac{Y-80}{8}$ has mean 0 and variance 1 and is approximately normal. Thus

   $$\mathbb{P}(0.25 < Y/400) = \mathbb{P}(Y > 100) = \mathbb{P}(\frac{Y-80}{8} > 2.5) \approx \mathbb{P}(Z > 2.5) = 0.9938.$$

8. (4.4.9) Let $f(x) = 1/x^2$ for $1 < x < \infty$, zero elsewhere, be the pdf of a random variable X. Consider a random sample of size 72 from the distribution (i.e. 72 i.i.d. random variables $X_1, \ldots, X_{72}$). Compute approximately the probability that more than 50 of the observations of the random sample are less than 3.

   Answer:

   Let Y denote the number of observations that are less than 3. Then $Y \sim Bin(72, p)$ where

$$p = \mathbb{P}(X_i < 3) = \int_1^3 \frac{1}{x^2} dx = \frac{2}{3}.$$

   Thus $\mathbb{E}[Y] = 48$ and $\mathrm{Var}(Y) = 16$ so $\sigma = 4$. Thus

$$\mathbb{P}(Y > 50) = \mathbb{P}(\frac{Y - 48}{4} > 0.5) = 1 - 0.6915$$