Math 361, Problem Set 2

September 17, 2010

Due: 9/13/10

1. (1.3.11) A bowl contains 16 chips, of which 6 are red, 7 are white and 3 are blue. If four chips are taken at random and without replacement, find the probability that

   (a) each of the 4 chips is red

   (b) none of the four chips is red

   (c) there is at least one chip of each color.

   Answer:

   For (a), there are $\binom{6}{4}$ ways to make such a choice so

   $$p = \frac{\binom{6}{4}}{\binom{16}{4}}$$

   For (b), the probability is $\frac{\binom{10}{4}}{\binom{16}{4}}$ (there are 10 non-red chips, of which I must choose 4).

   For (c), the probability is:

   $$p = \frac{\binom{6}{2}\binom{7}{1}\binom{3}{1} + \binom{6}{1}\binom{7}{2}\binom{3}{1} + \binom{6}{1}\binom{7}{1}\binom{3}{2}}{\binom{16}{4}}$$

   Note that this is equivalent to $\frac{1}{2} \times \frac{6 \cdot 7 \cdot 3 \cdot 13}{\binom{16}{4}}$.

2. (1.3.24) Consider three events C_1, C_2, C_3.

   (a) Suppose C_1, C_2, C_3 are mutually exclusive events. If $\mathbb{P}(C_i) = p_i$, for $i = 1, 2, 3$ what is the restriction on the sum p_1, p_2, p_3.

   (b) In the notation of Part (a), if $p_1 = 4/10$, $p_2 = 3/10$, and $p_3 = 5/10$ are C_1, C_2 and C_3 mutually exclusive?

   (c) Does it follow from your conclusion in part (b) that $\mathbb{P}(C_1 \cap C_2 \cap C_3) > 0$? Why or why not?

Answer:

For (a), $p_1 + p_2 + p_3 \leq 1$. For (b), this implies that the answer is no. However the answer to part (c) is also no - $\mathbb{P}(C_i \cap C_j) \geq 0$ for some pair, but all three need not intersect. For instance it is possible for $C_2 \subseteq C_1$ and $C_1 \cap C_2 = \emptyset$ while obtaining those probabilities.

3. (1.4.7) A pair of 6-sided dice is cast until either the sum of seven or eight appears.

    (a) Show that the probability of a seven before an eight is 6/11.

    (b) Next, this pair of dice is cast until a seven appears twice (as a sum) or until each of a six and an eight have appeared at least once. Show that the probability of the six and eight occuring before two sevens is 0.546.

    Answer: For (a) we ignore all rolls until we roll a seven or an eight. Now, considering the first roll where we roll a seven or an eight, what is the probability that it is a seven. Let R_i denote the probability that we roll an i. We want to compute:

$$\mathbb{P}(R_7 | R_8 \cup R_7) = \frac{\mathbb{P}(R_7)}{\mathbb{P}(R_7 \cup R_8)} = \frac{6/36}{11/36} = \frac{6}{11}.$$

For (b) consider the possibilies of ending with a 6 or an 8, only considering rolls where a 6,7, or 8 are rolled: We could get $68, 86, 678, 876, 768$ or 786. Clearly (since rolling 6 and 8 has the same probability) there is symmetry in the problem: we only need to compute the probability that we end because of a $68, 678 or 768$. Let $R_{6,8}$ (for instance) denote the event that I end because I rolled a 6 then an 8 with no sevens ever rolled. Then

$$\begin{aligned}
\mathbb{P}(R_{6,8}) &= \mathbb{P}(R_6 | R_6 \cup R_7 \cup R_8) \mathbb{P}(R_8 | R_7 \cup R_8) \\
&= \frac{\mathbb{P}(R_6)}{\mathbb{P}(R_6 \cup R_7 \cup R_8)} \cdot \frac{\mathbb{P}(R_8)}{\mathbb{P}(R_7 \cup R_8)} = \frac{5}{16} \cdot \frac{5}{11} = \frac{25}{176}.
\end{aligned}$$

as to reach that ending condition, I must have reached a 6 before a 7 or an 8, and then an 8 before a 7.

Likewise

$$\begin{aligned}
\mathbb{P}(R_{6,7,8}) &= \mathbb{P}(R_6 | R_6 \cup R_7 \cup R_8) \cdot \mathbb{P}(R_7 | R_7 \cup R_8) \mathbb{P}(R_8 | R_7 \cup R_8) \\
&= \frac{5}{16} \cdot \frac{6}{11} \cdot \frac{5}{11} = \frac{150}{1936} \\
\mathbb{P}(R_{7,6,8}) &= \mathbb{P}(R_7 | R_6 \cup R_7 \cup R_8) \cdot \mathbb{P}(R_6 | R_6 \cup R_7 \cup R_8) \mathbb{P}(R_8 | R_7 \cup R_8) \\
&= \frac{6}{16} \cdot \frac{5}{16} \cdot \frac{5}{11} = \frac{150}{2816}
\end{aligned}$$

Then

$$p = 2 \left(\frac{25}{176} + \frac{150}{1936} + \frac{150}{2816} \right) = \frac{4225}{7744} \approx 0.546.$$

4. (1.4.4) A hand of 13 cards is to be dealt at random and without replacement from an ordinary deck of playing cards. Find the conditional probability that there are at least three kings in the hand given that the hand contains two kings.

   Answer

   If I have at least 3 kings, I have either 3 or 4 kings, and if I have at least two kings I have exactly 2 or 3 or 4 kings. Let A denote the event that I have at least 3 kings, and B denote the event that I have at least 2.

   Then

$$\begin{aligned}\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} &= \frac{\mathbb{P}(A)}{\mathbb{P}(B)} = \frac{\frac{\binom{4}{3}\binom{48}{10}+\binom{4}{4}\binom{48}{9}}{\binom{52}{13}}}{\frac{\binom{4}{2}\binom{48}{11}+\binom{4}{3}\binom{48}{10}+\binom{4}{4}\binom{48}{9}}{\binom{52}{13}}} \\ &= \frac{\binom{4}{3}\binom{48}{10}+\binom{4}{4}\binom{48}{9}}{\binom{4}{2}\binom{48}{11}+\binom{4}{3}\binom{48}{10}+\binom{4}{4}\binom{48}{9}}\end{aligned}$$

5. (1.4.9) Bowl I contains 6 red chips and 4 blue chips. Five of these 10 chips are selected at random and without replacement and put into bowl II, which was initially empty. One chip is then drawn at random from bowl II. Given that this chip is blue, find the conditional probability that 2 red chips and 3 blue chips are transferred from bowl I to bowl II.

   Answer:

   Let B_i denote the event that i red chips were initially transferred, so that $\mathbb{P}(B_i) = \binom{6}{5-i}\binom{4}{i}/\binom{10}{5}$ for $i = 0, \ldots 4$, and let B denote the event that the final chip is blue, so that $p(B|B_i) = \frac{i}{5}$.

   We want to compute $\mathbb{P}(B_3|B)$. By Bayes rule:

$$\begin{aligned}\mathbb{P}(B_3|B) &= \frac{\mathbb{P}(B|B_3)\mathbb{P}(B_3)}{\sum \mathbb{P}(B|B_i)\mathbb{P}(B_i)} \\ &= \frac{\frac{3}{5}\binom{6}{2}\binom{4}{3}}{\frac{1}{5}\binom{6}{4}\binom{4}{1}+\frac{2}{5}\binom{6}{3}\binom{4}{2}+\frac{3}{5}\binom{6}{2}\binom{4}{3}+\frac{4}{5}\binom{6}{1}\binom{4}{4}}\end{aligned}$$

   (Note that when writing this out, I multiplied top and bottom by $\binom{10}{5}$, so that it'd be a bit simpler.)

6. (The two drunks problem): Two drunks, one stupid and one smart wander back to their apartment after a night of drinking to by stymied at their door. They both have n keys, but are unable to tell which one unlocks their apartment.

   (a) The first, stupid drunk picks a key from his keychain uniformly at random repeatedly *with replacement* (i.e. he lets the key fall back with the rest of his keys if it doesn't work). Let A_k be the event that he successfully opens the door on the kth try. Compute $\mathbb{P}(A_k)$ for all k.

(b) The smart drunk drops keys that don't work on the floor so he doesn't repeat bad keys. That is, he picks keys repeatedly *without replacement*. Let B_k be the event he successfully opens the door on the kth try. Compute $\mathbb{P}(B_k)$ for $k = 1, \ldots, n$. (Note: $\mathbb{P}(B_k) = 0$ for $k > n$. Why?)

Let P_k denote the event that the drunk picked the right key at time k. Note that, for (a), $A_k = P_1^c \cap P_2^c \cap \cdots \cap P_{k-1}^c \cap P_k$. Since $\mathbb{P}(P_i) = \frac{1}{n}$, and the P_i are independent we have that:

$$\mathbb{P}(A_k) = \mathbb{P}(P_1^c)\mathbb{P}(P_2^c)\ldots\mathbb{P}(P_{k-1}^c)\mathbb{P}(P_k) = \frac{1}{n}\left(\frac{n-1}{n}\right)^{k-1}.$$

For (b), the events P_i are no longer independent but:

$$\begin{aligned}
\mathbb{P}(A_k) &= \mathbb{P}(P_k|P_1^c \cap \cdots \cap P_{k-1}^c) \times \mathbb{P}(P_{k-1}^c|\mathbb{P}(P_1^c \cap \cdots \cap P_{k-2}^c) \times \cdots \\
&\quad \cdots \times \mathbb{P}(P_2^c|P_1^c)\mathbb{P}(P_1^c) \\
&= \frac{1}{n-(k-1)} \cdot \frac{n-(k-1)}{n-(k-2)} \cdots \frac{n-2}{n-1} \cdot \frac{n-1}{n} \\
&= \frac{1}{n}
\end{aligned}$$

for $k = 1, \ldots n$. That is, the door is equally likely to be opened at any point.