# Math 361, Problem set 5

Due 10/04/10

1. (1.6.8) Let $X$ have the pmf $p(x) = (\frac{1}{2})^x$, $x = 1, 2, 3, \ldots$, and zero elsewhere. Find the pmf of $Y = X^3$.

   *Answer:* $Y$ has pmf $p(x) = (\frac{1}{2})^{x/3}$ for $x = 1, 8, 27, \ldots$, and zero elsewhere.

2. (a) Pick a card from a standard deck. Let $X$ denote the rank of the card(counting ace as one, J=11, Q=12, K=13.) Find the pmf of $X$.

   (b) Pick two cards from a deck, with replacement. Let $Y$ denote the highest rank picked. Find the pmf of $Y$.

   *Answer:* Since all ranks are equally likely $X$ has pmf $p_X(x) = \frac{1}{13}$ for $x = 0, \ldots, 13$ and $p_X(x) = 0$ elsewhere.

   For $Y$; $p_(Y)(x) = \frac{1}{13^2} + 2\frac{n-1}{13^2}$ for $x = 0, \ldots, 13$ and $p_Y(x) = 0$ elsewhere. Here the $\frac{1}{13^2}$ represents both numbers being $x$, and the $2\frac{x-1}{13^2}$ represents picking the $x$ (with prob. $\frac{1}{13}$), picking the number less than $x$ (with prob. $\frac{x-1}{13}$), and considering the two possible orders.

   Note:

$$\sum_{n=1}^{13} \left( \frac{1}{13^2} + 2\frac{x-1}{13^2} \right) = \frac{1}{13} + \frac{2}{13^2} \cdot \frac{12 \cdot 13}{2} = \frac{1}{13} + \frac{12}{13} = 1.$$

3. (1.7.8) A mode of a distribution of one random variable $X$ is a value of $x$ that maximizes the pdf or pmf. For $X$ of the continuous type, $f(x)$ must be continuous. If there is only one such $x$, it is called the mode of the distribution. Find the mode of each of the following distributions:

   (a) $p(x) = (\frac{1}{2})^x$, $x, 1, 2, 3, \ldots$, zero elsewhere.

   (b) $f(x) = 12x^2(1-x), 0 < x < 1$, zero elsewhere.

   (c) $f(x) = \frac{1}{2}x^2e^{-x}$, $0 < x < \infty$, zero elsewhere.

   *Answer:* $\frac{1}{2^x}$ is decreasing, and hence maximized at $x = 1$. If $f(x) = 12x^2(1-x)$, then $f'(x) = 24x(1-x) - 12x^2$ which is maximized when $x = \frac{2}{3}$.

   If $f(x) = \frac{1}{2}x^2e^{-x}$, then $f'(x) = xe^{-x} - \frac{1}{2}x^2e^{-x}$ which is maximized when $x = 2$.

4. (1.7.14) Let $X$ have the pdf $f(x) = 2x$, $0 < x < 1$, zero elsewhere. Compute the probability that $X$ is at least $\frac{3}{4}$ given that $X$ is at least $\frac{1}{2}$.

   *Answer:*

$$\mathbb{P}(X \geq \frac{3}{4}|X \geq \frac{1}{2}) = \frac{\mathbb{P}(X \geq 3/4)}{\mathbb{P}(X \geq 1/2)} = \frac{\int_{3/4}^{1} 2xdx}{\int_{1/2}^{1} 2xdx} = \frac{7/16}{3/4} = \frac{7}{12}.$$

5. (1.7.17) Divide a line segment into two parts by selecting a point at random. Find the probability that the larger segment is at least 3 times the shorter. Assume the point is chosen uniformly.

   *Answer* Assume the line segment has length 1, and let $X$ denote the point on the segement where we split it. Then the larger segment is at least 3 times the shorter if $1 - x \geq 3x$ or $x \geq 3(1 - x)$. In other words; if $x \leq \frac{1}{4}$ or $x \geq \frac{3}{4}$. Since the line segment has length one, and we are choosing a point uniformly, this has probability $\frac{1}{2}$

6. (1.7.22) Let $X$ have the uniform pdf $f_X(x) = \frac{1}{\pi}$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Find the pdf of $Y = \tan(X)$. This is the pdf of a **Cauchy distribution**.

   *Answer* Here we have $g(x) = \tan(x)$, $g^{-1}(x) = \arctan(x)$ and $(g^{-1})'(x) = \frac{1}{1+x^2}$. Then

$$f_Y(y) = f_X(g^{-1}(y)) \cdot (g^{-1})'(y) = \frac{1}{\pi} \cdot \frac{1}{1+y^2}$$

   for $-\infty < y < \infty$. Here, the range of $y$ follows from the fact that the range of $\arctan(y)$ is $(-\pi/2, \pi/2)$ so $g^{-1}(y) \in (-\pi/2, \pi/2)$ for all $y \in \mathbb{R}$.