Math 361, Problem set 6

Due 10/18/10

1. (1.8.3) Let X have pdf $f(x) = (x+2)/18$ for $-2 < x < 4$, zero elsewhere. Find $\mathbb{E}[X], \mathbb{E}[(X+2)^3]$ and $\mathbb{E}[6X - 2(X+2)^3]$.

   Answer:

$$\mathbb{E}[X] = \frac{1}{18}\int_{-2}^{4} x^2 + 2x dx = \frac{1}{18}(\frac{x^3}{4} + x^2\Big|_{-2}^{4} = 2.$$

$$\mathbb{E}[(X+2)^3] = \frac{1}{18}\int_{-2}^{4}(x+2)^4 = \frac{1}{90}\ (x+2)^5|_{-2}^{4} = \frac{432}{5}.$$

$$\mathbb{E}[6X - 2(X+2)^3] = 6\mathbb{E}[X] - 2\mathbb{E}[(X+2)^3] = 12 - \frac{864}{5}.$$

2. (1.8.5) Let X be a number selected uniformly random from a set of numbers $\{51, \ldots, 100\}$. Approximate $\mathbb{E}[1/X]$. *Hint: Find reasonable upper and lower bounds by finding integrals bounding* $\mathbb{E}[1/X]$.

   Answer:

   Note that

$$\int_{50}^{100} \frac{1}{x} dx > \sum_{n=51}^{100} \frac{1}{n} > \int_{51}^{101} \frac{1}{x} dx.$$

   Thus

$$\mathbb{E}[1/X] = \frac{1}{50}\sum_{n=51}^{100}\frac{1}{n} \approx \frac{1}{100}(\int_{50}^{100}\frac{1}{x}dx + \int_{51}^{101}\frac{1}{x}dx) = \frac{1}{100}(\ln(100)+\ln(101)-\ln(50)-\ln(51)).$$

   Plenty of other reasonable estimates are available, eg.

$$\mathbb{E}[1/X] \approx \frac{1}{50}\int_{50.5}^{100.5}\frac{1}{x}dx.$$

3. Let X have the pdf $f(x) = 1/x^3$. Find $\mathbb{E}[X]$, but show that $e[X^2]$ does not exist.

Answer: This problem is badly misstated. To make it make sense, we can assume that $f(x) = 1/x^3$ on $x > \frac{1}{\sqrt{2}}$ so that the pdf integrates to 1. Then

$$\mathbb{E}[X] = \int_{1/\sqrt{2}}^{\infty} \frac{1}{x^2} dx = \sqrt{2}.$$

but

$$\mathbb{E}[X^2] = \int_{1/\sqrt{2}}^{\infty} \frac{1}{x} dx = \infty.$$

4. (1.8.14) Let X have the pdf $f(x) = 3x^2$, $0 < x < 1$, zero elsewhere.

   (a) Compute $\mathbb{E}[X^3]$

   (b) Show that $Y = X^3$ has a uniform(0,1) distribution.

   (c) Compute $\mathbb{E}[Y]$ and compare this result with the answer obtained in Part (a).

   Answer:

   We have

   $$\mathbb{E}[X^3] = \int_0^1 3x^5 dx = \frac{1}{2}.$$

   Note that if $Y = g(X)$ where $g(x) = x^3$. Thus $g^{-1}(y) = y^{1/3}$, and $(g^{-1})'(y) = \frac{1}{3}y^{-2/3}$. Thus

   $$f_Y(y) = f_X(g^{-1}(y))(g^{-1})'(y) = 3y^{2/3} \cdot \frac{1}{3}y^{-2/3} = 1,$$

   for $0 < y < 1$, zero else. Thus Y has a uniform(0,1) distribution.

   Thus

   $$\mathbb{E}[Y] = \int_0^1 y dy = \frac{1}{2}.$$

   and (of course) the answer to (a) and (c) are the same.

5. (1.9.4) If the $\mathbb{E}[X^2]$ exists, show that

   $$\mathbb{E}[X^2] \geq (\mathbb{E}[X])^2$$

   Answer:

   We have that

   $$0 \leq \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

   Rearranging gives the result. (*Note:* This is a special case of Jensen's inequality).

6. (1.9.8) Let X be a random variable such that $\mathbb{E}[(X-b)]$ exists for all real b. Show that $\mathbb{E}[(X-b)^2]$ is minimized when $b = \mathbb{E}[X]$.

   Answer:

   We have that

   $$\mathbb{E}[(X-b)^2] = \mathbb{E}[X^2 - 2bX + b^2] = \mathbb{E}[X^2] - 2b\mathbb{E}[X] + b^2.$$

   Differentiating with respect to b, we have

   $$\frac{d}{db}\mathbb{E}[(X-b)^2] = -2\mathbb{E}[X] + 2b.$$

   This has a critical point at $b = \mathbb{E}[X]$. This point is a minimum as $\frac{d}{db}\mathbb{E}[(X-b)^2]$ is positive if $b > \mathbb{E}[X]$ and negative if $b < \mathbb{E}[X]$.