# Math 361, Problem set 7

Due 10/25/10

1. (1.9.6) Let the random variable $X$ have $\mathbb{E}[X] = \mu$, $\mathbb{E}[(X-\mu)^2] = \sigma^2$ and mgf $M(t)$, $-h < t < h$. Show that

$$\mathbb{E}\left[\frac{X-\mu}{\sigma}\right] = 0, \qquad \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^2\right] = 1$$

and

$$\mathbb{E}\left[\exp\left(t\left(\frac{X-\mu}{\sigma}\right)\right)\right] = e^{-\mu t/\sigma} M\left(\frac{t}{\sigma}\right), \quad -h\sigma < t < h\sigma.$$

(Recall: $\exp(x) = e^x$).

*Answer:*

$$\mathbb{E}\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma}(\mathbb{E}[X] - \mu) = 0.$$

$$\mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^2\right] = \frac{1}{\sigma^2}\mathbb{E}[(X-\mu)^2] = \frac{\sigma^2}{\sigma^2} = 1.$$

$$\mathbb{E}\left[\exp\left(t\left(\frac{X-\mu}{\sigma}\right)\right)\right] = \mathbb{E}\left[\exp\left(\frac{tX}{\sigma}\right)\right] e^{-t\mu/\sigma} = M(t/\sigma)e^{-t\mu/\sigma},$$

so long as $M(t/\sigma)$ makes sense, which it does for $-h\sigma < t < h\sigma$.

2. (1.9.7) Show that the moment generating function of the random variable $X$ having pdf $f(x) = \frac{1}{3}$ for $-1 < x < 2$, zero elsewhere is

$$M(t) = \begin{cases} \frac{e^{2t} - e^{-t}}{3t} & t \neq 0 \\ 1 & t = 0 \end{cases}$$

*Answer:*

For $t \neq 0$

$$M(t) = \mathbb{E}[e^{tX}] = \int_{-1}^{2} e^{tx}\frac{1}{3}dx = \left.\frac{e^{tx}}{3t}\right|_{-1}^{2} = \frac{e^{2t} - e^{-t}}{3t}.$$

For $t = 0$,

$$M(t) = \mathbb{E}[e^{0X}] = \mathbb{E}[1] = 1.$$

3. (1.9.18) Find the moments of the distribution that has mfg $M(t) = (1 - t)^{-3}$, $t < 1$. *Hint:* Find the MacLaurin's series for $M(t)$.

   *Answer:*

   Taking derivatives starting with the geometric series, we have

$$\begin{aligned}
\frac{1}{1-t} &= \sum_{n=0}^{\infty} x^n \\
\frac{1}{(1-t)^2} &= \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n. \\
\frac{}{(1-t)^3} &= \frac{1}{2}\sum_{n=1}^{\infty} n(n+1)x^{n-1} = \sum_{n=0}^{\infty}(n+1)(n+2)x^n.
\end{aligned}$$

   By Taylor's theorem, the coefficient of $x^k$ satisfies

$$\frac{(k+1)(k+2)}{2} = c_k = \frac{M^{(k)}(0)}{k!}.$$

   Therefore

$$\mathbb{E}[X^k] = M^{(k)}(0) = \frac{(k+2)!}{2}.$$

4. (1.9.23) Consider $k$ continuous-type distributions with the following characteristics: pdf $f_i(x)$, mean $\mu_i$ and variance $\sigma_i^2$, $i = 1, 2, \ldots, k$. If $c_i \geq 0$, $i = 1, \ldots, k$ and $c_1 + \cdots + c_k = 1$, show that the mean and variance of the distribution having pdf $c_1 f_1(x) + \cdots + c_k f_k(x)$ are $\mu = \sum_{i=1}^{k} c_i \mu_i$, and $\sigma^2 = \sum_{i=1}^{k} c_i[\sigma_i^2 + (\mu_i - \mu)^2]$, respectively.

   *Answer:*

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \sum_{i=1}^{k} c_i f_i(x) dx = \sum_{i=1}^{k} c_i \int_{-\infty}^{\infty} x f_i(x) dx = \sum_{i=1}^{k} c_i \mu_i.$$

$$\begin{aligned}
\mathbb{E}[(X-\mu)^2] &= \int_{-\infty}^{\infty}(x-\mu)^2\sum_{i=1}^{k}c_i f_i(x)dx \\
&= \sum_{i=1}^{k}c_i\int_{-\infty}^{\infty}(x-\mu)^2 f_i(x)dx \\
&= \sum_{i=1}^{k}c_i\int_{-\infty}^{\infty}\big((x-\mu_i)^2+(\mu-\mu_i)(\mu+\mu_i-2x)\big)\,f_i(x)dx \qquad (*) \\
&= \sum_{i=1}^{k}c_i\left(\sigma_i^2+(\mu-\mu_i)(\mu+\mu_i-2\mu_i)\right) \\
&= \sum_{i=1}^{k}c_i\left(\sigma_i^2+(\mu-\mu_i)^2\right).
\end{aligned}$$

Where $(*)$ follows from the fact that:

$$(x-\mu)^2-(x-\mu_i)^2=\mu^2-\mu_i^2-2\mu x+2\mu_i x=(\mu-\mu_i)(\mu+\mu_i-2x)$$

5. (1.10.4) Let $X$ be a random variable with mgf $M(t)$, $-h<t<h$. Prove that

$$\mathbb{P}(X\ge a)\le e^{-at}M(t),\quad 0<t<h$$

and that

$$\mathbb{P}(X\le a)\le e^{-at}M(t),\quad -h<t<0.$$

Hint: Let $u(x)=e^{tx}$ and $c=e^{ta}$ in Markov's inequality (1.10.2)

*Answer:*

If $t>0$, then $u(x)=e^{tx}$ is positive, increasing and hence

$$\mathbb{P}(X\ge a)=\mathbb{P}(e^{tX}\ge e^{at})\le e^{-at}\mathbb{E}[e^{atX}]=e^{-at}M(t),$$

by Markov's inequality.

If $t<0$, then $y(x)=e^{tx}$ is positive, decreasing, and hence

$$\mathbb{P}(X\le a)=\mathbb{P}(e^{tX}\ge e^{at})\le e^{-at}M(t).$$

6. (1.10.3) If $X$ is a random variable such that $\mathbb{E}[X]=3$ and $\mathbb{E}[X^2]=13$, use Chebyshev's inequality to determine a lower bound for the probability $\mathbb{P}(-2<X<8)$.

*Answer*

Note $\sigma^2=\mathbb{E}[X^2]-\mathbb{E}[X]^2=4$. Then

$$\mathbb{P}(-2<X<8)=1-\mathbb{P}(|X-3|>5)=\mathbb{P}(|X-\mathbb{E}[X]|>5)\ge 1-\frac{\sigma^2}{5^2}=\frac{21}{25}.$$