Math 361, Problem set 9

Due 11/8/10

1. (2.2.3) Let X_1 and X_2 have the joint pdf $h(x_1, x_2) = 2e^{-x_1-x_2}$, $0 < x_1 < x_2 < \infty$, zero elsewhere. Find the joint pdf of $Y_1 = 2X_1$ and $Y_2 = X_2 - X_1$.
Answer: We have that $X_1 = Y_1/2$ and $X_2 = Y_2 + Y_1/2$. This gives us the Jacobian
$$J = \begin{vmatrix} 1/2 & 1 \\ 0 & 1 \end{vmatrix} = \frac{1}{2}.$$
Thus
$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{2} f_{X_1,X_2}(y_1/2, y_2+y_1/2) = exp^{-y_1-y_2} \quad 0 < y_1 < \infty \quad 0 < y_2 < \infty.$$

2. (2.3.2) Let $f_{1|2}(x_1|x_2) = c_1x_1/x_2^2$, $0 < x_1 < x_2$, $0 < x_2 < 1$ zero elsewhere, and $f_2(x_2) = c_2x_2^4$, $0 < x_2 < 1$, zero elsewhere, denote, respectively, te conditional pdf of X_1 given $X_2 = x_2$ and the marginal pdf of X_2. Determine
   (a) The constants c_1 and c_2.
   (b) The joint pdf of X_1 and X_2.
   (c) $\mathbb{P}(\frac{1}{4} < X_1 < 1/2|X_2 = \frac{5}{8})$
   (d) $\mathbb{P}(1/4 < X_1 < 1/2)$

   Answer: $f_{1|2}(x_1|x_2)$ is a pdf so:
$$1 = \int_0^{x_2} c_1x_1/x_2^2 dx_1 = \frac{c_1}{2}.$$
Thus $c_1 = 2$. Likewise, f_2 is a pdf, so $\int_0^1 c_2x_2^4 dx_2 = 1$, and hence $c_2 = 5$.
$f_{1,2}(x_1, x_2) = f_{1|2}(x_1|x_2)f_2(x_2) = 10x_1x_2^2$ for $0 < x_1 < x_2 < 1$.
For (c)
$$\mathbb{P}(\frac{1}{4} < X_1 < \frac{1}{2}|X_2 = \frac{5}{8}) = \int_{1/4}^{1/2} \frac{128}{25} x_1 dx_1 = \frac{12}{25}.$$

For (d)

$$\begin{aligned}
\mathbb{P}(\frac{1}{4} < X_1 < \frac{1}{2}) &= \int_{1/4}^{1/2}\int_{x_1}^{1} 10x_1x_2^2 dx_2 dx_1 \\
&= \int_{1/4}^{1/2} \frac{10}{3}(x_1^4 - x_1)dx_1 \\
&= \frac{449}{1536}.
\end{aligned}$$

3. (2.3.5) Let X_1 and X_2 be two random variables such that the conditional distributions and means exist. Show that

   (a) $\mathbb{E}[X_1 + X_2|X_2] = \mathbb{E}[X_1|X_2] + X_2$

   (b) $\mathbb{E}[u(X_2)|X_2] = u(X_2)$.

   Answer:

   First we check the identities conditioning on $X_2 = x_2$: that

   $$\begin{aligned}
   \mathbb{E}[X_1 + X_2|X_2 = x_2] &= \mathbb{E}[X_1|X_2 = x_2] + x_2 \quad \text{and} \\
   \mathbb{E}[u(X_2)|X_2 = x_2] &= u(x_2)
   \end{aligned}$$

   follows from linearity of expectation and the fact that X_2 and $u(X_2)$ are constants when we condition on $X_2 = x_2$. Since these identities hold *for every choice of* x_2, they hold for the general random variable X_2.

4. (2.3.9) Five cards are drawn and random and without replacement from an ordinary deck of cards. Let X_1 and X_2 denote, respectively, the number of spades and the number of hearts that appear in the five cards.

   (a) Determine the joint pmf of X_1 and X_2

   (b) Find the two marginal pmfs

   (c) What is the conditional pmf of X_2 given $X_1 = x_1$.

   Note: First two parts are similar to what was on your last homework! *Answer:*

   $$\mathbb{P}_{X_1,X_2}(x_1, x_2) = \frac{\binom{13}{x_1}\binom{13}{x_2}\binom{26}{5-x_1-x_2}}{\binom{52}{5}}$$

   so long as $x_1 + x_2 \leq 5$, 0 otherwise.

   For (b), we have that

   $$\mathbb{P}_{X_1}(x_1) = \frac{\binom{13}{x_1}\binom{39}{x_1}}{\binom{52}{5}} \quad \mathbb{P}_{X_2}(x_2) = \frac{\binom{13}{x_2}\binom{39}{x_2}}{\binom{52}{5}}.$$

   if $0 \leq x_1 \leq 5$ and $0 \leq x_2 \leq 5$; 0 otherwise. For (c) we have

   $$\mathbb{P}_{2|1}(x_2|x_1) = \frac{\binom{13}{x_2}\binom{26}{5-x_1-x_2}}{\binom{39}{x_2}}$$

if $0 \leq x_2 \leq 5 - x_1$.

5. (2.3.11) Let us choose at random a point from the interval $(0, 1)$ and let the random variable X_1 be equal to the number which corresponds to that point. Then choose a point at random from the interval $(0, x_1)$, where x_1 is the experimental value of X_1; and let the random variable X_2 be equal to the number which corresponds to this point.

   (a) Make assumptions about the marginal pdf $f_1(x_1)$ and the conditional pdf $f_{2|1}(x_2|x_1)$.

   (b) Compute $\mathbb{P}(X_1 + X_2 \geq 1)$.

   (c) Find the conditional mean $\mathbb{E}[X_1|x_2]$.

   Answer:

   We have $f_1(x_1) = 1$ for $0 < x_1 < 1$, zero otherwise, and $f_{2|1}(x_2|x_1) = \frac{1}{x_1}$, for $0 < x_2 < x_1$, 0 otherwise.

   For (b), we have that $f_{1,2}(x_1, x_2) = f_{2|1}(x_2|x_1) f_1(x_1) = \frac{1}{x_1}$ for $0 < x_2 < x_1 < 1$, and hence

$$\begin{aligned}\mathbb{P}(X_1 + X_2 \geq 1) &= \int_{1/2}^{1} + \int_{1-x_1}^{x_1} \frac{1}{x_1} dx_2 dx_1 \\ &= \int_{1/2}^{1} \frac{2x_1 - 1}{x_1} dx_1 = 1 - \ln(1) + ln(1/2) = 1 - ln(2).\end{aligned}$$

   For (c) we must first compute $f_{1|2}(x_1|x_2)$. We have

$$f_2(x_2) = \int_{x_2}^{1} \frac{1}{x_1} dx_1 = -\ln(x_2).$$

   Thus

$$f_{1|2}(x_1|x_2) = \frac{1}{x_1 \ln(1/x_2)}.$$

   So

$$\mathbb{E}[X_1|X_2 = x_2] = \int_{x_2}^{1} \frac{x_1}{x_1 \ln(1/x_2)} = \frac{1 - x_2}{\ln(1/x_2)}.$$