1. (*20 points*) There are two urns. The first has 3 red balls and 5 blue balls, the second has 6 red balls and 100 blue balls. A coin is flipped. If heads comes up a uniform random ball is picked from the first urn, if tails comes up a uniform random ball is picked from the second. Given that the picked ball is blue, what is the conditional probability that the coin was a heads.

H = event coin heads
B = event ball blue

Want:

$$\mathbb{P}(H \mid B) = \frac{\mathbb{P}(B\mid H)\mathbb{P}(H)}{\mathbb{P}(B\mid H)\,\mathbb{P}(H) + \mathbb{P}(B\mid T)\mathbb{P}(T)}$$

$$= \frac{\frac{5}{8}\cdot\frac{1}{2}}{\frac{5}{8}\cdot\frac{1}{2} + \frac{100}{106}\cdot\frac{1}{2}} = \frac{53}{133}$$

Okay answer, no need to simplify to

2. (*20 points*) A hand of 4 cards is picked from a standard deck. Compute

a. (*7 pts*) the probability that every card in the hand is of the same suit.

$$\frac{\binom{4}{1}\binom{13}{4}}{\binom{52}{4}}$$

Choose suit

choose cards

b. (*7 pts*) The hand contains exactly one pair.

$$\frac{\binom{13}{1}\binom{4}{2}\binom{12}{2}\binom{4}{1}\binom{4}{1}}{\binom{52}{4}}$$

Choose pair rank

Choose pair

Choose other ranks + cards

Note:

$$\frac{\binom{13}{1}\binom{4}{2}\binom{48}{2}}{\binom{52}{4}}$$

allows possibility of 2 pairs!

c. (*6 pts*) Every card in the hand is a different suit and a different rank.

Choose ♡ ♧ ◇ ♤

$$\frac{\binom{13}{1}\binom{12}{1}\binom{11}{1}\binom{10}{1}}{\binom{52}{4}}$$

Or

$$\frac{\binom{13}{4}\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{1}{1}}{\binom{52}{4}}$$

Choose ranks

Choose which is a heart, club, diamond, spade

3. (*20 points*) X is a continuous random variable with pdf $f(x) = 2e^{-2x}$ if $x \geq 0$, and $f(x) = 0$ otherwise. Compute

a. (*7 pts*) $\mathbb{P}(X \in C_1)$ where $C_1 = \mathbb{N} = \{1, 2, 3, 4, \ldots\}$.

$$\mathbb{P}(X=1) = \mathbb{P}(X=2) = \cdots 0 \quad \text{as} \quad \int_a^a 2e^{-2x}\,dx = 0.$$

$$\mathbb{P}(X \in C_1) = \sum_{k=1}^{\infty} \mathbb{P}(X=k) = 0.$$

b. (*7 pts*) $\mathbb{P}(X \geq 3)$

$$\int_3^\infty 2e^{-2x}\,dx = -e^{-2x}\Big|_3^\infty = e^{-6}$$

c. (*6 pts*) $\mathbb{P}(X \geq 3 | X \geq 2)$.

$$\mathbb{P}(X \geq 2) = \int_2^\infty 2e^{-2x}\,dx = -e^{-2x}\Big|_2^\infty = e^{-4},$$

$$\mathbb{P}(X \geq 3 | X \geq 2) = \frac{\mathbb{P}(X \geq 3)}{\mathbb{P}(X \geq 2)} = \frac{e^{-6}}{e^{-4}} = e^{-2}.$$

4. (*20 points*) Two fair dice are rolled.

a. (*10 pts*) Let X denote the absolute value of the difference between the rolls. Compute the pmf $p(x)$ of X.

x	$p(x)$
0	$6/36$
1	$10/36$
2	$8/36$
3	$6/36$
4	$4/36$
5	$2/36$
els	0

b. (*10 pts*) Determine the probability that the product of the two rolls is greater than the sum of the two rolls.

Easier: Product $\leq$ Sum: Only happens if at least one of rolls is one, or both are two.

$$\mathbb{P}(\text{Product} \leq \text{Sum}) = \frac{12}{36} = \frac{1}{3}$$

$(1,1)$ $(2,1)$ $(2,2)$
$\vdots$ $\vdots$
$(1,6)$ $(6,1)$

12 poss. w/ Product $\leq$ Sum.

$$\mathbb{P}(\text{Product} > \text{sum}) = \frac{2}{3}.$$

5. (*20 points*) **a.** (*10 pts*) Could $F(x) = \frac{1}{2}^x$ for $x = 0, 1, 2, 3 \ldots$ denote the cdf of a random variable X. Why or why not?

$F(x)$ is decreasing, could not be CDF.
(or, at least, not increasing)

b. (*10 pts*) A continuous random variable X has pdf $f(x) = \frac{1}{x^2}$ for $x \geq 1$, 0 otherwise. Compute the cdf and pdf of $Y = X^2$.

$Y = g(x) \quad g^{-1}(x) = \sqrt{x} \quad (g^{-1})'(x) = \frac{1}{2\sqrt{x}}$

$$f_Y(y) = f(g^{-1}(y))(g^{-1})'(y) = \begin{cases} \frac{1}{2y \cdot \sqrt{y}} & y \geq 1 \\ 0 & \text{else.} \end{cases}$$