1. (*20 points*) A coin is flipped until there are two heads in a row, or three tails. Find the probability that the three tails comes first.

Interpreting the problem as written

Method 1:

Find all possibilities: $\text{Prob} = 2^{-\text{length}}$

TTT   THTHT   HTHTHT

TTHT   HTTHT

THTT   HTHTT

HTTT

$$\mathbb{P}(3 \text{ tails before } 2 \text{ heads}) = \frac{1}{8} + 3 \cdot \frac{1}{16} + 3 \cdot \frac{1}{32} + 1 \cdot \frac{1}{64} = \frac{27}{64}$$

Method 2 Also: $\mathbb{P}(\text{Tail before } 2 \text{ heads}) = \frac{3}{4}$

$$\mathbb{P}(3 \text{ tails before } 2 \text{ heads}) = \left(\frac{3}{4}\right)^3 = \frac{27}{64}$$

If I asked for 2 heads in a row or 3 tails in a row

$$\mathbb{P}(T \mid H \cup T) = \frac{\mathbb{P}(T)}{\mathbb{P}(H \cup T)} = \frac{\frac{1}{8}}{\frac{1}{4} + \frac{1}{8}} = \frac{1}{3}$$

H = event 2 heads in a row

T = event 3 tails in a row

2. (*20 points*) Three integers are chosen with replacement from the first twenty integers. Find the probability that **a.** (*10 pts*) Their sum is even

Sum is even if 1) All are even

2) Two are odd, 1 even (3 possibilities)

$$\therefore \mathbb{P}\left(\begin{matrix}\text{Sum} \\ \text{even}\end{matrix}\right) = 4 \cdot \frac{1}{8} = \frac{1}{2}.$$

b. (*10 pts*) Their product is even.

Product is even unless all odd.

$$\mathbb{P}(\text{All odd}) = \frac{1}{8}.$$

$$\mathbb{P}(\text{Product even}) = 1 - \frac{1}{8} = \frac{7}{8}.$$

3. (*20 points*) X is a continuous random variable with cdf $F_X(x) = 1-(1-x)^2$ for $0 \le x \le 1$, with $F_X(x) = 0$ for $x \le 0$, and $F_X(x) = 1$ for $x \ge 1$. **a.** (*10 pts*) Find the pdf of X.

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} 1-(1-x)^2$$

$$F_X(x) \begin{cases} = 2(1-x) & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x \ge 1 \\ 0 & \text{for } x < 0. \end{cases}$$

b. (*10 pts*) Let $Y = 2(X+1)^2$. Find the cdf of Y.

$$Y = g(X) \quad g(x) = 2(x+1)^2$$
$$g^{-1}(x) = \sqrt{\frac{x}{2}} - 1$$

Then

$$F_Y(y) = \cancel{f}_X(g^{-1}(y)) = \begin{cases} 1-(1-(\sqrt{\frac{y}{2}}-1))^2 & 2 \le y \le 8 \\ 1 & y > 8 \\ 0 & y < 2. \end{cases}$$

4. (*20 points*) A random integer N from 1 to 10 is chosen uniformly at random.

a. (*10 pts*) The random variable X denotes the number of distinct prime factors of N. (So if $N = 8 = 2^3$, $X = 1$). Find the pmf of X.

X	$p(X)$
0	$1/10$
1	$7/10$
2	$2/10$
else	0.

1 ← 0 prime factors

2, 4, 8, 3, 9, 5, 7 ← 1 prime factor

6, 10 ← 2 prime factors

b. (*10 pts*) Another random integer N' from 1 to 10 is also chosen uniformly at random. Find the probability that $N > 2N'$.

Possible (N, N'):

$(*, 1)$: $* = 3 - 10$: 8 possibilities

$(*, 2)$: $* = 5 - 10$: 6 possibilities

$(*, 3)$: $* = 7 - 10$: 4 poss.

$(* \, 4)$: $* = 9 - 10$ = 2 poss

$$\therefore \; P(N > 2N') = \frac{20}{100} = \frac{1}{5}.$$

5. (*20 points*) Is it possible for two events A and B with $\mathbb{P}(A), \mathbb{P}(B) > 0$ to be both mutually exclusive and independent. Why or why not?

If ~~P(A)~~ A, B are independent

$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) > 0$, so

no way for A and B to be mutually exclusive (that is for $\mathbb{P}(A \cap B) = \emptyset$).