1. (*20 points*) **a.** (*10 pts*) 10 people are in a room. Assuming their birthdays are independent and uniformly distributed, what is the probability that at least two of them share a birthday.

$$1 - \frac{365 \cdot 364 \cdot 363 \cdot \cdots \cdot 356}{365^{10}}$$

b. (*10 pts*) How many people must be in a room so that the probability that at least two of them share a birthday is 0.5?

Want:

$$1 - \frac{365 \cdot 364 \cdot \cdots (365-k+1)}{365^k} \geq \frac{1}{2}$$

happens when $k \geq 23$.

2. (*20 points*) Recall: The mode of a random variable X is the value of x where the pmf $p(x)$ or pdf $f_X(x)$ is maximized (depending on whether the random variable is continuous.) Find the mode of the following random variables.

a. (*8 pts*) X is discrete with pmf $p(x) = e^{-10}\frac{10^x}{x!}$ for $x = 0, 1, 2, 3, \ldots$.

$$\frac{p(x+1)}{p(x)} = \frac{\frac{e^{-10}10^{x+1}}{(x+1)!}}{\frac{e^{-10}10^{x}}{x!}} = \frac{10}{x+1}.$$

$p(x)$ increasing until $x=9$

$p(9)=p(10)$

then $p(x)$ decreasing.

Mode $= x=9$ or $x=10$.

b. (*7 pts*) X is continuous with pdf $f_X(x) = \frac{\sin(x)}{2}$, $0 \leq x \leq \pi$ $f_X(x) = 0$ otherwise.

need to make a pdf

$$\frac{d}{dx} f_X(x) = \frac{\cos x}{2}, \quad \text{zero at } x = \frac{\pi}{2}.$$

Mode is $x = \frac{\pi}{2}$

c. (*7 pts*) X is continuous with pdf $f_X(x) = \frac{1}{e-1}e^x$ for $0 \leq x \leq 1$, $f_X(x) = 0$ otherwise.

$\frac{1}{e-1}e^x$ is increasing.

Mode is $x=1$.

3. (*20 points*) **a.** (*10 pts*) Construct an example showing that 3 events can be pairwise independent, but not mutually independent.

Ex From class:

Flip 3 coins

A: First Flip heads

B = ~~S~~econd Flip heads

C = Both Flips same

$$\mathbb{P}(A \cap B) = \mathbb{P}(A \cap C) = \mathbb{P}(B \cap C) = \frac{1}{4}$$

But $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A \cap C) = \frac{1}{4}$
$$\neq$$
$$\frac{1}{8}$$

b. (*10 pts*) Suppose A, B, and C are mutually independent with $\mathbb{P}(A) = 1/2$, $\mathbb{P}(B) = 1/4$, $\mathbb{P}(C) = 3/8$. Find

$$\mathbb{P}((A \cup B^c) \cap C^c)$$

$$\mathbb{P}((A \cup B^c) \cap C^c) = \mathbb{P}(A \cup B^c)\,\mathbb{P}(C^c)$$
$$= (\mathbb{P}(A) + \mathbb{P}(B^c) - \mathbb{P}(A)\mathbb{P}(B^c))\,\mathbb{P}(C^c)$$
$$= \left(\frac{1}{2} + \frac{3}{4} - \frac{3}{8}\right)\left(\frac{5}{8}\right) = \frac{35}{64}$$

4. (*20 points*) A random word is chosen unformly from the sentence

"How much wood could a woodchuck chuck if a woodchuck could chuck wood."

Let X denote the length of the word.

a. (*10 pts*) Find the pmf of X

x	$p(x)$
1	2/13
2	1/13
3	1/13
4	3/13
5	4/13
6	0
7	0
8	0
9	2/13
else	0

b. (*10 pts*) Suppose two words are chosen uniformly from the sentence, with replacement. What is the probability that they have the same length?

$$\mathbb{P}(\text{Same length}) = \underbrace{\left(\frac{2}{13}\right)^2}_{\text{both } 1} + \underbrace{\left(\frac{1}{13}\right)^2}_{\text{both } 2} + \left(\frac{1}{13}\right)^2 + \left(\frac{3}{13}\right)^2 + \left(\frac{4}{13}\right)^2 + \left(\frac{2}{13}\right)^2$$

Note: Without replacement is

$$\underbrace{\left(\frac{2}{13}\right)\left(\frac{1}{12}\right)}_{\text{both } 1} + \underbrace{\left(\frac{3}{13}\right)\left(\frac{2}{12}\right)}_{\text{both } 4} + \underbrace{\left(\frac{4}{13}\right)\left(\frac{3}{12}\right)}_{\text{both } 5} + \underbrace{\left(\frac{2}{13}\right)\left(\frac{1}{12}\right)}_{\text{both } 9}$$

5. (*20 points*) **a.** (*10 pts*) Suppose X is discrete with pmf $p(x) = \frac{6}{\pi^2} \cdot \frac{1}{x^2}$, $p(x) = 0$ otherwise. Find the pmf of $Y = X^2$.

Should be for $x = 1, 2, 3, \ldots$

$Y = g(X)$ where $g^{-1}(X) = \sqrt{X}$

$$P_Y(y) = p_X(\sqrt{y}) = \begin{cases} \frac{6}{\pi^2} \cdot \frac{1}{y} & \text{for } y = 1, 4, 9, 16, \ldots \\ 0 & \text{else.} \end{cases}$$

b. (*10 pts*) Suppose X if continuous with pdf $f(x) = \frac{1}{x^2}$ for $x \geq 1$, $f(x) = 0$ otherwise. Find the pdf of $Y = X^2$.

$g^{-1}(X) = \sqrt{X}$ $\quad (g^{-1})'(x) = \frac{1}{2\sqrt{X}}$

$$f_Y(y) = f_X(g^{-1}(y)) \cdot (g^{-1})'(y) = \begin{cases} \frac{1}{2y^{3/2}} & y \geq 1 \\ 0 & \text{else} \end{cases}$$