1. (*20 points*) Suppose random variables X_1, X_2 and X_3 have joint pdf $f(x_1, x_2, x_3) = \frac{2(x_1+x_2+x_3)}{3}$ for $0 < x_1 < 1$, $0 < x_2 < 1$, $0 < x_3 < 1$.

a. (*10 pts*) Suppose $Y_1 = X_1$, $Y_2 = X_1 + X_2$, $Y_3 = X_1 + X_2 + X_3$. Compute the joint pdf of Y_1, Y_2, Y_3.

$$X_1 = Y_1$$
$$X_2 = Y_2 - Y_1$$
$$X_3 = Y_3 - Y_2$$

$$J = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 1.$$

$$f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3) = \frac{y_1 + (y_2 - y_1) + (y_3 - y_2)}{3} \cdot 1 = \frac{y_3}{3}$$

for $\{ 0 < y_1 < 1,\ y_1 < y_2 < y_1 + 1,\ y_2 < y_3 < y_2 + 1$; 0 else

For:
$$0 < y_1 < 1$$
$$0 < y_2 - y_1 < 1 \implies$$
$$0 < y_3 - y_2 < 1$$

$$0 < y_1 < 1$$
$$y_1 < y_2 < y_1 + 1$$
$$y_2 < y_3 < y_2 + 1$$

b. (*10 pts*) Compute $\mathbb{E}[Y_2]$.

$$\mathbb{E}[Y_2] = \int_0^1 \int_0^1 \int_0^1 (x_1 + x_2) \frac{(x_1 + x_2 + x_3)}{3} \, dx_1 \, dx_2 \, dx_3$$

$$= \int_0^1 \int_0^1 \int_0^1 \frac{x_1^2 + 2x_1x_2 + x_2^2 + x_1x_3 + x_2x_3}{3} \, dx_1 \, dx_2 \, dx_3$$

$$= \int_0^1 \int_0^1 \frac{\frac{1}{3} + x_2 + x_2^2 + \frac{1}{2}x_3 + x_2x_3}{3} \, dx_2 \, dx_3$$

$$= \int_0^1 \frac{\frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{2}x_3 + \frac{1}{2}x_3}{3} \, dx_3$$

$$= \frac{\frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{2}}{3} = \boxed{\frac{5}{9}}$$

Alt:

$$\int_0^1 \int_{y_1}^{y_1+1} \int_{y_2}^{y_2+1} \frac{y_2 y_3}{3} \, dy_3 \, dy_2 \, dy_1 = \cdots = \frac{5}{9}$$

2. (*20 points*) Prescott Pharmaceuticals has developed a new test for the disease Horrible type A. Only $\frac{1}{1000}$ people is infected with Horrible type A, and the test is correctly positive on sick people 98% of the time. Unfortunately 2% of the time, it returns positive on a well person. Let S denote the event a person is sick, and P denote the event a test is positive.

a. (*10 pts*) Determine $\mathbb{P}(S|P)$.

Bayes:

$$\mathbb{P}(S|P) = \frac{\mathbb{P}(P|S)\mathbb{P}(S)}{\mathbb{P}(P|S)\mathbb{P}(S) + \mathbb{P}(P|\bar{S})\mathbb{P}(\bar{S})}$$

$$= \frac{.98\left(\frac{1}{1000}\right)}{.98\left(\frac{1}{1000}\right) + .02\left(\frac{999}{1000}\right)} = 0.0467\ldots$$

b. (*10 pts*) Suppose instead, the false positive rate was p instead of 2% (with all other parameters equal). How small must p be so that $\mathbb{P}(S|P) = .99$?

W/ prob p:

$$.99 = \mathbb{P}(S|P) = \frac{.98\left(\frac{1}{1000}\right)}{.98\left(\frac{1}{1000}\right) + p\left(\frac{999}{1000}\right)}$$

Solve for p:

$$p\left(\frac{999}{1000}\right) = \frac{.98}{.99}\left(\frac{1}{1000}\right) - .98\left(\frac{1}{1000}\right)$$

$$p = \frac{1}{999}(.98)\left(\frac{1}{.99} - 1\right) \approx 9.9 \times 10^{-6}$$

3. (*20 points*) A uniformly random variable X is chosen in $(0,1)$. Then $X_1, X_2, X_3, \ldots$ are independent random variables chosen uniformly in $(0, X)$. Let $Y_n = \max\{X_1, \ldots, X_n\}$. Show that $Y_n \to_p X$.

X_n uniform on $(0, X)$

so $\mathbb{P}(|X_n - X| > \varepsilon) = \dfrac{X-\varepsilon}{X} = 1 - \dfrac{\varepsilon}{X} < 1 - \varepsilon,$

$\therefore \mathbb{P}(\cancel{\max} |Y_n - X| > \varepsilon) = \mathbb{P}(|X_1 - X| > \varepsilon, |X_2 - X| > \varepsilon, \ldots |X_n - X| > \varepsilon)$

$\leq (1-\varepsilon)^n \to 0$

as $n \to \infty$ if $\varepsilon > 0$.

$\therefore Y_n \to_p X.$

4. (*20 points*)

a. (*10 pts*) Jim didn't study for his multiple choice probability test and is uniformly and independently randomly choosing one of the four answers for each of the 200 problems. Let X denote his percentage (/100) on the exam. Estimate

$$\mathbb{P}(X > 30).$$

$\cancel{X}$ = # questions correct. $Y \sim Bin(200, \frac{1}{4})$

$\mathbb{E}[Y] = np = 50$

$Var(Y) = n(p)(1-p) = \frac{150}{4}$

$\sigma = \sqrt{\frac{150}{4}} \simeq 6.123$

$$P(X>30) = \mathbb{P}(Y>60)$$

$$\mathbb{P}(Y>60) = \mathbb{P}\left(\frac{Y-50}{\sigma} > \frac{60-50}{6.123}\right)$$

$$\approx \mathbb{P}(Z > 1.63299)$$

$$\approx 1 - .9474 = 0.0526$$

b. (*10 pts*) A lightbulbs life in years is given by the exponential distribution $f(x) = \frac{1}{2}e^{-x/2}$. Mary buys a pack of 64 lightbulbs at Costco, and uses them one by one. Let X_i denote the life of the ith bulb, then their total life is $Y = \sum_{i=1}^{64} X_i$. Estimate

$$\mathbb{P}(Y < 110)$$

$\mathbb{E}[X_i] = 2 \left(= \frac{1}{\lambda}\right)$

$Var(X_i) = 4 = \left(\frac{1}{\lambda^2}\right)$

$\sigma = 2$

$\frac{Y - n\mu}{\sqrt{n}\sigma} \simeq N(0,1)$ distributed by CLT.

$$\therefore P(Y<110) = P\left(\frac{Y-128}{16} < \frac{110-128}{16}\right)$$

$$\approx \mathbb{P}\left(Z < -\frac{18}{16}\right) = \mathbb{P}(Z < -1.125)$$

$$\approx 1 - .8686$$

$.8686 = \mathbb{P}(Z \le 1.12)$

Note: I'd accept: $1 - .8708$ $(= \mathbb{P}(Z \le 1.13))$ or the average here too!

$$\boxed{= .1314}$$

5. (*20 points*) A hand of 6 cards are drawn from a deck without replacement.

a. (*10 pts*) Find the pmf for the number of clubs in the hand.

$$p_X(x) = \frac{\binom{13}{x}\binom{39}{6-x}}{\binom{52}{6}} \quad \text{for } x = 0, \dots, 6$$

$$0 \text{ otherwise}$$

b. (*10 pts*) Find the probability that the hand contains a four of a kind (four cards of the same rank - no restrictions on the other two cards).

$F \sim$ event of four of a kind

$$\mathbb{P}(5) = \frac{\underbrace{\binom{13}{1}}_{\text{Choose rank for 4 of a kind}}\underbrace{\binom{48}{2}}_{\text{Choose other 2 cards}}}{\binom{52}{6}}$$

6. (*20 points*)

a. (*10 pts*) Show that $\text{Var}(\alpha X) = \alpha^2 \text{Var}(X)$.

$$\text{Var}(\alpha X) = \mathbb{E}[(\alpha X)^2] - (\mathbb{E}[\alpha X])^2$$
$$= \mathbb{E}[\alpha^2 X^2] - (\alpha\, \mathbb{E}[X])^2$$
$$= \alpha^2 \mathbb{E}[X^2] - \alpha^2 \mathbb{E}[X]^2$$
$$= \alpha^2(\mathbb{E}[X^2] - \mathbb{E}[X]^2) = \alpha^2 \text{Var}(X)$$

b. (*10 pts*) Show that $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$ if X and Y are independent.

$$\text{Var}(X+Y) = \mathbb{E}[(X+Y)^2] - \mathbb{E}[X+Y]^2$$
$$= \mathbb{E}[X^2 + 2XY + Y^2] - (\mathbb{E}[X] + \mathbb{E}[Y])^2$$
$$= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - (\mathbb{E}[X]^2 + 2\mathbb{E}[X]\mathbb{E}[Y] + \mathbb{E}[Y]^2)$$
$$= \underbrace{\mathbb{E}[X^2] - \mathbb{E}[X]^2}_{\text{Var}(X)} + \underbrace{\mathbb{E}[Y^2] - \mathbb{E}[Y]^2}_{\text{Var}(Y)} + 2(\underbrace{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}_{=0,\text{ as } X, Y \text{ indep}})$$
$$= \text{Var}(X) + \text{Var}(Y)$$

7. (*20 points*) A dice is rolled until a 6 is rolled or two fives (not necessarily in a row) are rolled. Find the probability that the two fives come first.

[redacted] A = event 6 before 5, B = event 6 before 2 fives.

$\mathbb{P}(A) = \frac{1}{2}$ $\mathbb{P}(B|\bar{A}) = \frac{1}{2}$

[redacted] $\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B|\bar{A})\,\mathbb{P}(\bar{A}) = \frac{3}{4}$, so

$\mathbb{P}(\bar{B}) = \frac{1}{4}$, where $\bar{B}$ is the event 2 fives come first.

8. (*20 points*) A dice is rolled, let X be the result of that roll. Then X cards are drawn from a deck, let Y denote the number of hearts in that deck.

a. (*10 pts*) Determine

$$\mathbb{P}_{Y|X}(y|x)$$

(Hint: No need to find $p_{X,Y}(x,y)$ to do this part.)

$$\mathbb{P}_{Y|X}(y|x) = \begin{cases} \dfrac{\binom{13}{y}\binom{39}{x-y}}{\binom{52}{x}} & \text{for } y = 0, \dots x \\ 0 & \text{otherwise} \end{cases}$$

b. (*10 pts*) Determine

$$\mathbb{P}_{X,Y}(x,y).$$

(Hint: Use part (a).)

$$\frac{\mathbb{P}_{X,Y}(x,y)}{\mathbb{P}_X(x)} = \mathbb{P}_{Y|X}(y|x),$$ but we know $\mathbb{P}_X(x) = \frac{1}{6}$ for $x = 1, \dots, 6$.

$$\therefore\ \mathbb{P}_{X,Y}(x,y) = \mathbb{P}_X(x)\,\mathbb{P}_{Y|X}(y|x) = \begin{cases} \dfrac{1}{6}\,\dfrac{\binom{13}{y}\binom{39}{x-y}}{\binom{52}{x}} & x = 1, \dots 6,\ y = 0, \dots X \\ 0 & \text{else.} \end{cases}$$