Midterm Exam II

Math 361
9/27/10

Name: ______________________

Read all of the following information before starting the exam:

- READ EACH OF THE PROBLEMS OF THE EXAM CAREFULLY!
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- A single 8 1/2 $\times$ 11 sheet of notes (double sided) is allowed. Calculators are permitted.
- Circle or otherwise indicate your final answers.
- Please keep your written answers clear, concise and to the point.
- This test has . problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Turn off cellphones, etc.
- READ EACH OF THE PROBLEMS OF THE EXAM CAREFULLY!
- Good luck!

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$\sum$	

1. (*20 points*) Thirteen cards are drawn at random without replacement from a standard deck. Let X denote the number of face cards drawn (kings, queens or jacks) and let Y denote the number of spades drawn.

a. (*10 pts*) Find the marginal pmfs of X and Y.

$$p_X(x) = \begin{cases} \dfrac{\binom{12}{x}\binom{40}{13-x}}{\binom{52}{13}} & \text{for } x = 0, 1, \ldots, 12 \\ 0 & \text{else} \end{cases}$$

$$P_Y(y) = \begin{cases} \dfrac{\binom{13}{y}\binom{39}{13-y}}{\binom{52}{13}} & \text{for } y = 0, 1, \ldots, 13 \\ 0 & \text{else} \end{cases}$$

b. (*10 pts*) Find the joint pmf $p_{X,Y}(x, y)$.

$$P_{X,Y}(x,y) = \begin{cases} \dfrac{\binom{10}{y}\binom{9}{x}\binom{30}{13-y-x} + \binom{3}{1}\binom{10}{y-1}\binom{9}{x-1}\binom{30}{13-x-y+1} + \binom{3}{2}\binom{10}{y-2}\binom{9}{x-2}\binom{30}{13-x-y+2} + \binom{3}{3}\binom{10}{y-3}\binom{9}{x-3}\binom{30}{13-x-y+3}}{\binom{52}{13}} & \text{for } x = 0, \ldots 12,\; y = 0, \ldots 13,\; x+y \le 16 \\ 0 & \text{else} \end{cases}$$

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Note: 16, not 13 because they can overlap.

here $\binom{n}{k} = 0$ if $k < 0$.

Could write:

$$P_{X,Y}(x,y) = \begin{cases} \dfrac{\sum_{i=0}^{3} \binom{3}{i}\binom{10}{y-i}\binom{9}{x-i}\binom{30}{13-x-y+i}}{\binom{52}{13}} & x = 0, \ldots 12,\; y = 0, \ldots 13 \\ 0 & \text{else} \end{cases}$$

Note:

3 = # of spades that are face cards

9 = # of non-spades face cards

10 = # of non-face card spades

30 = # of non-face card non-spades.

2. (*20 points*) **a.** (*10 pts*) Suppose X_n has the Gamma distribution $\Gamma(n, \beta)$. Explain why

$$\frac{X_n - \frac{n}{\beta}}{\frac{\sqrt{n}}{\beta}} \to_d N(0,1).$$

X_n has Γ dist, so

$X_n = \sum_{i=1}^{n} Y_i$ where Y_i are expo(β).

Thus this follows by CLT.

(No need to use MGF method: If I say

Show $X_n \to_d N(0,1)$: use MGF (or other) method

Explain $X_n \to_d N(0,1)$: can appeal to CLT.

b. (*10 pts*) Prove the weak law of large numbers: If X_n are iid with $\mathbb{E}[X] = \mu$ and $\text{Var}(X) = \sigma^2 < \infty$, let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then

$$\bar{X}_n \to_p \mu$$

$\mathbb{E}[\bar{X}_n] = \mathbb{E}\left[\frac{1}{n}\sum X_i\right] = \mu$

$\text{Var}(\bar{X}_n) = \text{Var}\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n^2}\text{Var}\left(\sum X_i\right) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}.$

Need to show: $P(|\bar{X}_n - \mu| > \varepsilon) \to 0$ for all $\varepsilon > 0$. 10

This is true because:

$P(|\bar{X}_n - \mu| > \varepsilon) \leq \frac{\text{Var}(\bar{X}_n)}{\varepsilon^2}$ by Chebychev

$= \frac{\sigma^2}{n\varepsilon^2} \to 0$ as $n \to \infty$.

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3. (*20 points*) **a.** (*10 pts*) Let X, Y and Z be random variables with joint pdf $f(x, y, z) = e^{-x-y-z}$ for $x > 0, y > 0, z > 0$. Compute

$$\mathbb{P}(X < Y < Z | Z > 1). = \frac{P(X<Y<Z \cap Z>1)}{\mathbb{P}(Z>1)}$$

$$\mathbb{P}(X<Y<Z, Z>1) = \int_{z=1}^{\infty}\int_{0}^{z}\int_{0}^{y} e^{-x-y-z}\,dx\,dy\,dz = \int_{z=1}^{\infty}\int_{0}^{z} e^{-z-y} - e^{-z-2y}\,dy\,dz$$

$$= \int_1^\infty \left(e^{-z} - e^{-2z}\right) - \frac{1}{2}\left(e^{-z} - e^{-3z}\right) dz$$

$$= \int_1^\infty \frac{1}{2}e^{-z} - e^{-2z} + \frac{1}{2}e^{-3z}\,dz = \frac{1}{2e} - \frac{1}{2e^2} + \frac{1}{6e^3}$$

$$P(Z>1) = \int_1^\infty\int_0^\infty\int_0^\infty e^{-x-y-z}\,dx\,dy\,dz = e^{-1}$$

$$\therefore \frac{\mathbb{P}(X<Y<Z, Z>1)}{\mathbb{P}(Z>1)} = \frac{\left(\frac{1}{2e} - \frac{1}{2e^2} + \frac{1}{6e^3}\right)}{1/e} = \boxed{\frac{1}{2} - \frac{1}{2e} + \frac{1}{6e^2}}$$

b. (*10 pts*) Are X, Y, Z independent? Explain.

$$f_X(x) = \begin{cases} \int_0^\infty\int_0^\infty e^{-x-y-z}\,dy\,dz = e^{-x} & x \geq 0 \\ 0 & \text{else.} \end{cases}$$

By symmetry $f_Y(y) = \begin{cases} e^{-x} & y \geq 0 \\ 0 & \text{else} \end{cases}$ and $f_Z(z) = \begin{cases} e^{-z}, & z \geq 0 \\ 0 & \text{else} \end{cases}$

Since $f(x, y, z) = f_X(x)\, f_Y(y)\, f_Z(z)$

X, Y, Z are independent.

4. (*20 points*) **a.** (*10 pts*) Is it possible for two events A and B to be independent and mutually exclusive? Why or why not?

It is possible so long as $\mathbb{P}(A)=0$ or $\mathbb{P}(B)=0$.

Then $\mathbb{P}(A\cap B)=0$ and $\mathbb{P}(A)\mathbb{P}(B)=0$,

If $\mathbb{P}(A)>0$ and $\mathbb{P}(B)>0$

then it is impossible,

as $\mathbb{P}(A\cap B)=0<\mathbb{P}(A)\mathbb{P}(B)$.

b. (*10 pts*) Independent events A, B and C have $\mathbb{P}(A)=0.9$, $\mathbb{P}(B)=0.3$ and $\mathbb{P}(C)=0.2$. Find

$$\mathbb{P}((A\cup B)\cap C^c)$$

$= \mathbb{P}(A\cup B)\mathbb{P}(C^c) = (\mathbb{P}(A)+\mathbb{P}(B)-\mathbb{P}(A\cap B))\,\mathbb{P}(C^c)$

$= (\mathbb{P}(A)+\mathbb{P}(B)-\mathbb{P}(A)\mathbb{P}(B))\,\mathbb{P}(C^c)$

$= (1.2-.27)(.8)$.

5. (*20 points*) A random angle X is chosen according to pdf $f_X(x) = \frac{2}{\pi^2}x$ for $0 < x < \pi$, 0 otherwise. Let $Y = \cos(X)$.

a. (*10 pts*) Compute $\mathbb{E}[Y]$.

$$\mathbb{E}[Y] = \int_0^{\pi} \cos(x) \cdot \frac{2}{\pi^2} x \, dx = \frac{2}{\pi^2}\left(x\sin x \Big|_0^{\pi} - \int_0^{\pi} \sin x \, dx \right)$$

$u = x$ $\quad dv = \cos(x)$

$du = dx$ $\quad v = \sin(x)$

$$= \frac{2}{\pi^2}\left(+\cos(x)\Big|_0^{\pi} \right) = -\frac{4}{\pi^2}$$

b. (*10 pts*) Find the pdf $f_Y(y)$.

Note: $\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}$.

$Y = g(X)$ $\quad g(t) = \cos(x)$ ← $\underline{\text{decreasing}}$ function for $0 < x < \pi$

$g^{-1}(x) = \arccos(x)$

$(g^{-1})'(x) = -\frac{1}{\sqrt{1-x^2}}$

So $f_Y(y) = -f_X(g^{-1}(y))(g^{-1})'(y) = \begin{cases} \frac{2}{\pi^2} \frac{\arccos(y)}{\sqrt{1-y^2}} & \text{for } -1 < y < 1. \\ 0 & \text{else} \end{cases}$

g decreasing!

6. (*20 points*) **a.** (*10 pts*) Suppose X is a $N(2,3)$ random variable. Determine

$$\mathbb{P}(X > 1).$$

$$\mathbb{P}(X>1) = \mathbb{P}\left(\frac{X-2}{\sqrt{3}} > \frac{1-2}{\sqrt{3}}\right)$$

$$= \mathbb{P}(Z > -.5773)$$

$$\approx \mathbb{P}(Z > -.58) = .7190$$

b. (*10 pts*) Y is a $Bin(320, \frac{1}{8})$ random variable. Estimate $\mathbb{P}(X < 35)$.

P $E[Y] = 40$

$Var(Y) = np(1-p) = 35$

$$\mathbb{P}(X<35) = \mathbb{P}\left(\frac{X-40}{\sqrt{35}} < \frac{35-40}{\sqrt{35}}\right)$$

$$= \mathbb{P}(Z < -.8451..)$$

$$\approx \mathbb{P}(Z < -.85) = 1 - .8023$$

$$= .1977$$

7. (*20 points*) There are three widget factories construction widgets. The first constructs 1000 in a day, with 5% defective. The second constructs 10000 in a day, but only 1% are defective. The third constructs 4000 a day, but 10% are defective.

a. (*10 pts*) One of the 15000 widgets constructed today is chosen uniformly at random. It is defective. What is the probability it came from the second factory?

F_i = event came from ith factory $\quad D$ = event defective

$$P(F_i|D) = \frac{P(D|F_i)P(F_i)}{P(D)}$$

$$= \frac{\frac{100}{10000}\cdot\frac{10000}{15000}}{\frac{550}{15000}}$$

$$= \frac{100}{550} = \frac{2}{11}$$

50 defective from plant 1

100 " " " 2

400 " " " 3

b. (*10 pts*) How many widgets must be selected at random (with replacement) from the 15000 constructed today so that the probability of finding a defective one is at least 0.5?

N_k = event none defective when choosing k

$$\mathbb{P}(N_k) = \left(\frac{14450}{15000}\right)^k$$

Want $\mathbb{P}(N_k) < \frac{1}{2}$, so $\left(\frac{14450}{15000}\right)^k < \frac{1}{2}$

$$k \ln\left(\frac{14450}{15000}\right) < \ln\left(\frac{1}{2}\right)$$

$$k > \frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{14450}{15000}\right)} = 18.55\ldots$$

$k \geq 19$ suffices.

8. (*20 points*)
Random variables X and Y have conditional pdf $f_{X|Y}(x|y) = e^{-x+2y}$ for $x > 2y$ and Y has pdf $f_Y(y) = 2e^{-2y}$ for $y \geq 0$.

a. (*10 pts*) Find $\mathbb{E}[X|y]$.

$$\mathbb{E}[X|y] = \int_{2y}^{\infty} x e^{-x+2y}\,dx = e^{2y}\left((1+2y)e^{-2y}\right)$$

$$= 1+2y.$$

b. (*10 pts*) Compute the pdf of $\mathbb{E}[X|Y]$.

$$\mathbb{E}[X|Y] = 1+2Y.$$

$$= g(Y) \quad \text{where} \quad g(y) = 1+2y$$

$$g^{-1}(y) = \frac{y-1}{2} \qquad (g^{-1})'(y) = \frac{1}{2}$$

$$f_{\mathbb{E}[X|Y]}(y) = f_Y(g^{-1}(y))\,(g^{-1})'(y)$$

$$= \begin{cases} e^{-(y-1)} & \text{for } y \geq \\ 0 & \text{else.} \end{cases}$$

9. (*20 points*) Three integers are chosen with replacement from the first twenty integers. Find the probability that **a.** (*8 pts*) Their sum is even.

$P(\text{Sum even}) = \frac{1}{2}$

↳ If the sum of the first two is even, the sum is even so long as the last one is even (which has prob. $\frac{1}{2}$)

If the sum of the first two is odd, the sum is ~~odd~~ even so long as the last one is odd (which has prob $\frac{1}{2}$).

b. (*7 pts*) Their product is even

$P(\text{product even}) = 1 - \left(\frac{1}{2}\right)^3 = \frac{7}{8}$

the product is even unless all 3 are odd:

c. (*5 pts*) Their product is at least 10.

Possibilities w/ product < 10?

~~1,1,1~~

$1,1,*$ $\quad * = 1,..9$

$1,2,*$ $\quad * = 1,..4$

$1,3,*$ $\quad * = 1,..3$

$1,4,*$ $\quad * = 1,..2$

$1,5,1$ ⋮ $1,9,1$ } 5 poss.

23 poss.

$2,1,*$ $\quad * = 1,..4$

$2,2,*$ $\quad * = 1,2$

$2,3,1$, $2,4,1$ } 2 poss

8 poss

$3,1,*$ $\quad * = 1,2,3$

$3,*,1$ $\quad * = 2,3$

5 po

$4,1,1$, $4,1,2$, $4,2,1$ } 3 poss.

$5,1,1$ ↓ $9,1,1$

5 poss

Total of:

$22 + 8 + 5 + 3 + 4$

$= 42$ possibilities.

$P(\text{Prod} \geq 10) = 1 - P(\text{Prod} < 10)$

$= 1 - \frac{44}{(20)^3}$.