**1.** (*20 points*) A random variable $X$ has pdf $f(x) = \frac{1}{2}e^{-|x|}$ for $-\infty < x < \infty$ **a.** (*8 pts*) Compute Var$(X)$ and $\mathbb{E}[X]$.

$$\boxed{E[X] = \int_{-\infty}^{\infty} x e^{-|x|} \cdot \tfrac{1}{2}\, dx = 0} \quad \text{(odd function)}$$

$$E(X^2) = \int_{-\infty}^{\infty} \tfrac{1}{2} x^2 e^{-|x|} dx = \int_0^{\infty} x^2 e^{-x} dx = -x^2 e^{-x}\Big|_0^{\infty} + 2\int_0^{\infty} x e^{-x} dx = 2$$

$$\text{Var}(X) = E[X^2] - E[X]^2 = 2 - 0 = 2$$

**b.** (*6 pts*) Use Chebyschev's inequality to estimate $\mathbb{P}(|X| > 100)$.

$$\mathbb{P}(|X| > 100) \leq \frac{\sigma^2}{100^2} = \frac{2}{100^2}$$

**c.** (*6 pts*) Compute $\mathbb{P}(|X| > 100)$.

$$\mathbb{P}(|X| > 100) = 2\int_{100}^{\infty} \tfrac{1}{2} e^{-x} dx = -e^{-x}\Big|_{100}^{\infty}$$

$$\boxed{= e^{-100}.}$$

Note: $e^{-100}$ is much less than $\frac{2}{100^2}$, so here Chebyschev is pretty poor.

**2.** (*20 points*) A dice is rolled 5 times. Let $X$ denote the maximum value rolled, and let $Y$ denote the minimum value rolled. **a.** (*10 pts*) Compute the marginal pmf of $X$ $p_X(x)$.

$$p_X(x) = \left(\frac{x}{6}\right)^5 - \left(\frac{x-1}{6}\right)^5 \qquad x = 1 \dots 6.$$

#'s all $\leq x$ ~~less than~~

but not all $< x$

**b.** (*10 pts*) Compute the joint pmf $p_{X,Y}(x,y)$.

$$\le x, y \le 6.$$

$$p_{X,Y}(x,y) = \begin{cases} 0 & x < y \\ \left(\frac{1}{6}\right)^5 & x = y \\ \left(\frac{x-y+1}{6}\right)^5 - 2\left(\frac{x-y}{6}\right)^5 + \left(\frac{x-y-1}{6}\right)^5 & \text{else} \end{cases}$$

#'s all between x & y

but not all strictly $> y$ or strictly $< x$

over counted possibilities w/ #'s $> y$ and $< x$

$$\int x^2 e^{-x} dx = x^2 e^{-x} + 2\int x e^{-x} = x^2 e^{x} - 2xe^{x} + 2e^{x} \qquad \int xe^{x} = xe^{x} - e^{x}$$

**3.** (*20 points*) Let $X$ and $Y$ have joint pdf $f_{X,Y}(x,y) = e^{-x-y}$ for $0 < x < y < \infty$.
a. (*7 pts*) Compute $\mathbb{E}[X^2 + 2YX]$.

$$E[X^2+2YX] \cancel{E[X^2]} = \int_{y=0}^{\infty}\int_{x=0}^{y} (x^2+2xy)\, e^{-x-y}\, dx\, dy$$

$$\cancel{= \int_{y=0}^{\infty} (y^2 +}$$

See other page. Sorry it's a mess.

b. (*7 pts*) Compute the conditional pdf $f_{X|Y}(x|y)$.

$$f_Y(y) = \int_0^y e^{-x-y} dx = -e^{-x-y}\Big|_0^y \qquad 0<y<\infty$$

$$= e^{-y} - e^{-2y}$$

$$f_{X|Y}(x|y) = \frac{e^{-x-y}}{e^{-y}-e^{-2y}} \qquad 0<x<y$$

c. (*6 pts*) Compute the condition expectation $\mathbb{E}[X|Y = y]$.

$$E[X|y] = \int_0^y \frac{xe^{-x-y}}{e^{-y}-e^{-2y}} dx = \frac{e^{-y}}{e^{-y}-e^{-2y}}\left(\int_0^y xe^{-x} dx\right)$$

$$= \frac{e^{-y}}{e^{-y}-e^{-2y}}\left(-xe^{-x} - e^{-x}\Big|_0^y\right)$$

$$= \frac{e^{-y}}{e^{-y}-e^{-2y}}\left(-ye^{-y} - e^{-y} + 1\right)$$

**4.** (*20 points*) Suppose $X_1$ and $X_2$ have joint pdf $f_{X_1,X_2}(x,y) = \frac{3}{8}x^3y$ for $0 < x < y < 2$. Let $Y_1 = X_1X_2$ and $Y_2 = X_2$. **a.** (*10 pts*) Compute the joint pdf of $Y_1$ and $Y_2$

$\cancel{f_{X_1,X_2}(x,y)}$

$Y_1 = X_1X_2 \qquad X_1 = \frac{Y_1}{Y_2} \qquad X_2 = Y_2 \qquad J = \begin{vmatrix} \frac{1}{y_2} & y_1 \\ 0 & 1 \end{vmatrix} = \frac{1}{Y_2}$

$$\boxed{f_{X_1,X_2}(y_1,y_2) = \frac{3}{8}\left(\frac{y_1}{y_2}\right)^3 y_2 \cdot \frac{1}{y_2} = \frac{3}{8}\left(\frac{y_1}{y_2}\right)^3 \quad 0 \text{ else}}$$

for

Bounds $\boxed{\begin{array}{l} 0 < y_1 < y_2^2 \\ \cancel{0 < y_2} \\ \sqrt{y_1} < y_2 < 2. \end{array}}$

$0 < \frac{y_1}{y_2} < y_2 < 2$

$0 < y_1 < y_2^2$

**b.** (*10 pts*) Compute the marginal pdf of $Y_1$.

$$f_{Y_1}(y_1) = \int_{y_2=\sqrt{y_1}}^{2} \frac{3}{8}\left(\frac{y_1}{y_2}\right)^3 dy_2 = \frac{3}{8}y_1^3\left(\frac{y_2^{-2}}{-2}\right)\Bigg|_{\sqrt{y_1}}^{2}$$

$$\boxed{f_{Y_1}(y) = \frac{3}{8}y_1^3\left(\frac{1}{2y_1} - \frac{1}{8}\right) \qquad 0 < y_1 < 4. \qquad 0 \text{ else.}}$$

**5.** (*20 points*) Suppose $X$ has pdf $f_X(x) = xe^{-x}$ for $x \geq 0$. **a.** (*10 pts*) Compute the mgf of $X$, $M_X(t)$. For what $t$ does this exist?

$$M_X(t) = \mathbb{E}[e^{tx}] = \int_0^\infty x e^{(t-1)x}\,dx$$

$$= \frac{xe^{(t-1)x}}{(t-1)} \; \cancel{+} \; \frac{e^{(t-1)x}}{(t-1)^2}\Bigg|_0^\infty$$

$$= \frac{1}{(t-1)^2} \text{ assuming } t<1.$$

$M_X(t)$ exists if $|t|<1$.

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**b.** (*10 pts*) $Y$ has ~~mgf~~ mgf $M_Y(t) = \frac{1}{1+2t}$. Compute $\mathbb{E}[Y^3]$.

$$M_Y(t) = \frac{1}{1+2t} = 1 - 2t + 4t^2 - 8t^3 + \cdots$$

$$M_Y(t) = \sum \frac{\mathbb{E}[Y^n]}{n!} t^n$$

$$\therefore \; -8 = \frac{\mathbb{E}[Y^3]}{3!} \quad \text{so} \quad \mathbb{E}[Y^3] = -48.$$

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$$E[X^2+2YX] = \int_0^\infty \int_0^y (x^2+2xy)e^{-x-y}\,dx\,dy = \int_0^\infty e^{-y}\int_0^y (x^2+2xy)e^{-x}\,dx\,dy$$

$$= \int_0^\infty e^{-y}\left(-x^2e^{-x} - 2xe^{-x} - 2e^{-x} + 2y(-xe^{-x} - e^{-x})\right)\Big|_0^y\,dy$$

$$= \int_0^\infty e^{-y}\left(-y^2e^{-y} - 2ye^{-y} - 2e^{-y} - 2y^2e^{-y} - 2ye^{-y} + 2 + 2y\right)dy$$

$$= \int_0^\infty -3y^2e^{-2y} - 4ye^{-2y} - 2e^{-2y} + 2e^{-y} + 2ye^{-y}\,dy$$

$$= -3\left(\frac{-y^2e^{-2y}}{2} - \frac{ye^{-2y}}{2} - \frac{e^{-2y}}{4}\Big|_0^\infty\right) - 4\left(\frac{ye^{-2y}}{-2} - \frac{e^{-2y}}{4}\Big|_0^\infty\right)$$

$$-2\left(\frac{e^{-2y}}{-2}\Big|_0^\infty\right) + 2 + 2$$

$$= -3\left(\frac{1}{4}\right) - 4\left(\frac{1}{4}\right) - 2\left(\frac{1}{2}\right) + 4$$

$$= 2 - \frac{3}{4} = \boxed{\frac{5}{4}}$$

**1.** (*20 points*) Ten cards are drawn without replacement from a deck of 52 cards. $X$ denote the number of red cards and $Y$ denotes the number of clubs.

**a.** (*7 pts*) Compute the marginal pdf $f_X(x)$

$$f_X(x) = \begin{cases} \dfrac{\binom{26}{x}\binom{26}{10-x}}{\binom{52}{10}} & x = 0, \ldots, 10. \\[2ex] 0 & \text{else} \end{cases}$$

**b.** (*7 pts*) Compute the joint pdf $f_{X,Y}(x, y)$.

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{\binom{26}{x}\binom{13}{y}\binom{13}{10-x-y}}{\binom{52}{10}} & 0 \le x+y \le 10 \\[2ex] 0 & \text{else} \end{cases}$$

**c.** (*6 pts*) Compute the conditional pdf $f_{X|Y}(x|y)$.

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \qquad f_Y(y) = \frac{\binom{13}{y}\binom{39}{10-y}}{\binom{52}{10}}$$

$$= \frac{\binom{26}{x}\binom{13}{10-x-y}}{\binom{39}{10-y}} \qquad 0 \le x \le 10-y;$$

$0$ else

**2.** (*20 points*) $X$ and $Y$ are continuous random variables with $f_{X,Y} = 9e^{-x}y^{-10}$ for $0 < x < \infty$, $1 < y < \infty$ and 0 otherwise. **a.** (*10 pts*) Compute $\mathbb{E}[X]$, and $\mathbb{E}[XY]$

$$\mathbb{E}[X] = \int_{y=1}^{\infty}\int_{x=0}^{\infty} 9xe^{-x}y^{-10}\,dx\,dy = \int_{1}^{\infty} 9y^{-10}\,dy = -y^{-9}\Big|_1^{\infty} = \underline{1}$$

using: $\int_0^{\infty} xe^{-x}\,dx$.

$$\mathbb{E}[XY] = \int_{y=1}^{\infty}\int_{x=0}^{\infty} 9xe^{-x}y^{-9}\,dx\,dy = \int_{1}^{\infty} 9y^{-9}\,dy = -\frac{9}{8}y^{-8}\Big|_1^{\infty}$$

$$\boxed{= \frac{9}{8}}$$

**b.** (*10 pts*) Suppose $Z_1 = X + Y$ and $Z_2 = Y$. Find the joint pdf $f_{Z_1,Z_2}(z_1, z_2)$.

$Z_1 = X + Y \qquad Z_2 = Y$

$X = Z_1 - Z_2$

$Y = Z_2$

$$J = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

$$f_{Z_1 Z_2}(z_1, z_2) = 9e^{-(z_1 - z_2)}z_2^{-9} \quad \text{for } 1 < z_2 < z_1 < \infty.$$

for: $0 < z_1 - z_2 < \infty \Rightarrow z_1 > z_2$

$1 < z_2 < \infty$

**3.** (*20 points*) $X$ is a random variable with $\mathbb{E}[X] = 100$ and $\mathrm{Var}(X) = 20$. Let $Y = 10X$.

**a.** (*10 pts*) Estimate $\mathbb{P}(X < 0 \text{ or } X > 200)$.

$$\mathbb{P}(X<0 \text{ or } X>200) = \mathbb{P}(|X - \underset{\mu}{\overset{\shortparallel}{100}}| \geq 100) \leq \frac{\sigma^2}{100^2} = \frac{20}{100^2}$$

**b.** (*10 pts*) Let $Y = 10X$. Compute $\mathbb{E}[Y]$ and $\mathrm{Var}(Y)$.

$$\mathbb{E}[10X] = 10\,\mathbb{E}[X] = 1000$$

$$\mathrm{Var}(10X) = \mathbb{E}[(10X)^2] - \mathbb{E}[10X]^2$$

$$= \mathbb{E}[100X^2] - 100\,\mathbb{E}[X]^2$$

$$= 100(\mathrm{Var}(X)) = 2000.$$

Note: $\mathrm{Var}(kX) = k^2\,\mathrm{Var}(X)$.

constant

**5.** (*20 points*) We choose a random variable $X$ uniformly in $(0,1)$ and then a random variable $Y$ uniformly in the interval $(X, 1)$. **a.** (*8 pts*) Find $f_X(x)$ and $f_{Y|X}(y|x)$.

$$f_X(x) = 1 \quad 0<x<1, \quad 0 \text{ else}$$

$$f_{Y|X}(y|x) = \frac{1}{1-x} \quad x<y<1, \quad 0 \text{ else}$$

**b.** (*6 pts*) Find the joint pdf $f_{X,Y}(x, y)$.

$$f_{X,Y}(x,y) = f_X(x)\, f_{Y|X}(y|x)$$

$$= \begin{cases} \frac{1}{1-x} & 0<x<y<1 \\ 0 & \text{else} \end{cases}$$

**c.** (*6 pts*) Compute $\mathbb{E}[X|Y = y]$

$$f_Y(y) = \int_0^y \frac{1}{1-x}\,dx = -\ln(1-y) \qquad 0<y<1$$

$$f_{X|Y}(x|y) = \frac{1}{(1-x)\left(\ln\left(\frac{1}{1-y}\right)\right)} \qquad 0<x<y$$

$$\mathbb{E}[X|Y=y] = \int_0^y \frac{x}{(1-x)\ln\left(\frac{1}{1-y}\right)}\,dx = \frac{1}{\ln\left(\frac{1}{1-y}\right)} \int_0^y \left(-1 + \frac{1}{1-x}\right) dx$$

$$= \frac{1}{\ln(1/1-y)}\left(-y - \ln(1-y)\right)$$

$$\int x e^{tx}\,dx = \frac{\cancel{x} e^{tx}}{\cancel{t}} - \frac{1}{t}\int e^{tx}\,dx = \frac{xe^{tx}}{t} - \frac{1}{t^2}e^{tx}.$$

**4.** (*20 points*) **a.** (*10 pts*) Suppose $X$ and $Y$ have joint pdf $f(x) = 6xy$ for $0 < x < y < 1$, zero otherwise. Compute the joint moment generating function $M_{X,Y}(t_1, t_2)$. For what $t_1, t_2$ does this exist?

$$\cancel{E[e^{t_1X+t_2Y}] = \int_{y=0}^{1}\int_{0}^{y} e^{t_1x+t_2y}\, 6xy\,dx\,dy = \int_0^1 6ye^{t_2y}\left(\frac{xe^{tx}}{t} - \right.}$$

Far too long: See scrap page

**b.** (*10 pts*) A random variable $X$ has moment generating function $M(t) = \frac{e^t}{2-e^t}$. Compute $\mathrm{Var}(X)$.

$\mathrm{Var}(X) = E[X^2] - E[X]^2$

$E[X] = M'(0) = 2$

$E[X^2] = M^{(2)}(0)$

$= 6.$

$$\boxed{\mathrm{Var}(X) = 6 - 4 = 2}$$

$M(t) = e^t(2-e^t)^{-1}$

$M'(t) = e^t(2-e^t)^{-1} + e^{2t}(2-e^t)^{-2}$

$M'(0) = 2 = E[X]$

$M''(t) = e^t(2-e^t)^{-1} + 3e^{2t}(2-e^t)^{-2}$

$+ 2e^{3t}(2-e^t)^{-3}$

$M''(0) = 6 = E[X^2]$

$$\int x^2 e^{tx}dx = \frac{x^2e^{tx}}{t} - \int \frac{2xe^{tx}}{t}dx = \frac{x^2e^{tx}}{t} - \frac{2xe^{tx}}{t^2} + \frac{2e^{tx}}{t^3}$$

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$$\int xe^{tx} = \frac{xe^{tx}}{t} - \frac{e^{tx}}{t^2}$$

$$E\left[e^{t_1X+t_2Y}\right] = \int_{y=0}^{1}\int_0^y e^{t_1x+t_2y}\,6xy\,dx\,dy$$

$$= \int_{y=0}^{1} 6ye^{t_2y}\left(\frac{xe^{t_1x}}{t_1} - \frac{e^{t_1x}}{t_1^2}\Big|_0^y\right)dy$$

$$= \int_{y=0}^{1} 6y^2\frac{e^{(t_1+t_2)y}}{t_1} - 6y\frac{e^{(t_1+t_2)y}}{t_1^2} + \frac{6ye^{t_2y}}{t_1^2}\,dy.$$

$$= \left(\frac{6}{t_1}\left(\frac{y^2e^{(t_1+t_2)y}}{t_1+t_2} - 2\frac{ye^{(t_1+t_2)y}}{(t_1+t_2)^2} + 2\frac{e^{(t_1+t_2)y}}{(t_1+t_2)^3}\right)\right.$$

$$- \frac{6}{t_1^2}\left(\frac{ye^{(t_1+t_2)y}}{t_1+t_2} - \frac{e^{(t_1+t_2)y}}{(t_1+t_2)^2}\right)$$

$$+ \frac{6}{t_1^2}\left(\frac{ye^{t_2y}}{t_2} - \frac{e^{t_2y}}{t_2^2}\right)\Bigg|_{y=0}^{1}$$

$$\boxed{\begin{aligned} = &\frac{6}{t_1}\left(e^{(t_1+t_2)}\left(\frac{1}{t_1+t_2} - \frac{2}{(t_1+t_2)^2} + \frac{2}{(t_1+t_2)^3}\right) - \frac{2}{(t_1+t_2)^3}\right) \\ &- \frac{6}{t_1^2}\left(e^{t_1+t_2}\left(\frac{1}{t_1+t_2} - \frac{1}{(t_1+t_2)^2}\right) + \frac{1}{(t_1+t_2)^2}\right) \\ &+ \frac{6}{t_1^2}\left(e^{t_2}\left(\frac{1}{t_2} - \frac{1}{t_2^2}\right) + \frac{1}{t_2^2}\right) = M(t_1,t_2) \end{aligned}}$$

for all $|t_1|<\infty$, $|t_2|<\infty$

Sorry!