Practice Midterm Exam I

Math 362 **Name:** ______________________________

2/25/10

Read all of the following information before starting the exam:

- READ EACH OF THE PROBLEMS OF THE EXAM CAREFULLY!
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- A single 8 1/2 $\times$ 11 sheet of notes (double sided) is allowed. Calculators are permitted.
- Copies of normal, t-distribution and χ^2 tables are at the back
- Circle or otherwise indicate your final answers.
- Please keep your written answers clear, concise and to the point.
- This test has . problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Turn off cellphones, etc.
- READ EACH OF THE PROBLEMS OF THE EXAM CAREFULLY!
- Good luck!

Practice 1 Solns

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$\sum$	

1. (*20 points*) A sample $X_1, X_2, \ldots, X_n$ is drawn from the pdf $f(x;\theta) = \frac{2x}{\theta^2}$ for $0 < x < \theta$. Let $Y_1 < Y_2 < \ldots < Y_n$ denote the order statistics of the sample.

a. (*5 pts*) Find the pdf of Y_n.

$$f_{Y_n}(y_n) = n\,F_X(y_n)^{n-1} f(y_n) = 2n \frac{y_n^{2n-1}}{\Theta^{2n}} \quad \boxed{\text{for } 0 < y_n < \Theta}$$

$$F_X(y_n) = \int_0^{y_n} \frac{2x}{\Theta^2}\,dx = \frac{y_n^2}{\Theta^2}$$

b. (*7 pts*) Find c that makes cY_n an unbiased estimator for θ.

$$E[Y_n] = \int_0^{\Theta} 2n \frac{y_n^{2n}}{\Theta^{2n}} = \frac{2n}{2n+1} \frac{y_n^{2n+1}}{\Theta^{2n}} \Big|_0^{\Theta} = \frac{2n}{2n+1}\Theta$$

$\therefore \frac{2n+1}{2n} Y_n$ is an unbiased estimator for Θ

eg. $\boxed{c = \frac{2n+1}{2n}}$

c. (*8 pts*) Suppose $n = 5$, and $\theta = 1$. Find the probability that Y_n is a potential outlier for its distribution.

$$f_{Y_5}(y) = 10\,y^9 \quad 0<y<1 \qquad F_{Y_5}(y) = y^{10} \quad 0<y<1.$$

$\tilde{Q}_1 = 6$ st $F_{Y_5}(6) = .25$, $6^{10} = .25 \Rightarrow Q_1 = \left(\frac{1}{4}\right)^{1/10} \approx .87$

$Q_3 = 6$ st $F_{Y_5}(6) = .75 \Rightarrow Q_3 = \left(\frac{3}{4}\right)^{1/10} = .97$

$\therefore h = 1.5(Q_3 - Q_1) = .15$

X potential outlier if $X < .87 - .15$ or $X > .97 + .15$ (impossible) so

$\boxed{\text{Prob. of potential outlier is } P(Y_n < .72) = .037}$

2. (*20 points*) $X_1, \ldots, X_n$ are independent Poisson random variables with mean λ. We wish to test

$$H_0 : \lambda = \frac{1}{10} \quad vs \quad H_1 : \lambda > \frac{1}{10}.$$

a. (*10 pts*) Suppose $n = 10$. We accept H_1 if $Y = \sum X_i > C$. How large must C be in order for the size of the test to be less than .05?

Size of test $= P_{\frac{1}{10}}(Y > C)$ $\quad Y \sim \text{Pois}(1)$.

Want: $1 - P_{\frac{1}{10}}(Y > C) \geq .95$ $\quad \boxed{\text{Take } C = 2}$

$\Downarrow$

$P_{\frac{1}{10}}(Y \leq C) \geq .95$

$C=2$: $P_{\frac{1}{10}}(Y \leq C) = e^{-1} + e^{-1} + \frac{e^{-1}}{2} \approx .92$ so $C=2$ not enough,

but:

$C=3$: $P_{\frac{1}{10}}(Y \leq C) = e^{-1}\left(1 + 1 + \frac{1}{2} + \frac{1}{6}\right) \approx .98 > .95$, so $\boxed{C=3}$ enough

b. (*10 pts*) Another possible test is the following: We accept H_1 if more than 1/10 of the observed values are non-zero. Find the *approximate size* of this test. You may leave your answer in terms of Φ, the cdf of the standard normal distribution and n. Do not assume $n = 10$.
Hint: The number of non-zero X_i's is binomially distributed.

Let $Y =$ # of non-zero's.

$P_{\frac{1}{10}}(X_i \neq 0) = 1 - P_{\frac{1}{10}}(X_i = 0) = 1 - e^{-1/10} =: p \approx .095$

$Y \sim \text{Bin}(n, p)$. if $\lambda = \frac{1}{10}$

Want: $P_{\frac{1}{10}}\left(Y > \frac{n}{10}\right)$

$= P_{\frac{1}{10}}\left(Y - pn > \left(\frac{1}{10} - p\right)n\right)$

$= P_{\frac{1}{10}}\left(\frac{Y - pn}{\sqrt{np(1-p)}} > \frac{\left(\frac{1}{10} - p\right)n}{\sqrt{np(1-p)}}\right)$

$\sim N(0,1)$ by Central Limit Theorem

$\approx 1 - \Phi\left(\frac{\left(\frac{1}{10} - p\right)}{p(1-p)} \sqrt{n}\right) \approx 1-\Phi\left(\frac{(.005)}{(.095)(.905)} \sqrt{n}\right)$

3. (*20 points*) 100 Emory students are asked their favorite course, the distribution of their answers is as follows:

Subject	ENG 317	TBT 102	MATH 362	PE 168	CHEM 350
Freq	10	15	50	20	5

Our initial impression of the popularities of the courses (in the order presented above) is $q_0 = .1, q_1 = .1, q_3 = .55, q_4 = .15, q_5 = .1$. We wish to test

$$H_0 : q_0 = .1, \quad q_1 = .1, \quad q_3 = .56, \quad q_4 = .14, \quad q_5 = .1 \qquad vs \qquad H_1 : \text{all other possibilities}$$

Using the chi-square test, will we accept H_0 or H_1 at the 0.05 confidence level? At the 0.1 confidence level?

We compute:

$$\sum \left(\frac{(X_i - q_i n)^2}{q_i n}\right) = \left(\frac{(10-10)^2}{10}\right) + \left(\frac{(15-10)^2}{10}\right) + \frac{(50-56)^2}{56} + \frac{(20-14)^2}{14}$$

$$+ \frac{(5-10)^2}{10} = 8.214.$$

Since $\sum \frac{(X_i - q_i n)^2}{q_i n} \approx \chi^2(4)$ if our q_i are correct, we look at our table in the back.

Since $\mathbb{P}(\chi^2(k-1) > 9.488) = 0.05$

and $8.214 < 9.488$, we accept H_0 at .05 confidence level.

Since $P(\chi^2(k-1) > 7.779) = 0.1$,

we accept H_1 at .1 confidence level.

4. (*20 points*) A sample of 20 $N(\mu, \sigma)$ random variables, where μ and σ are both unknown, is generated with $\bar{X} = 13.27$ and $S^2 = 24.15$.

a. (*10 pts*) Construct a 99%-confidence interval for μ. *Note:* Construct a real, not approximate, confidence interval.

We know:

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t \text{ dist w/ } \overset{(n-1)}{19} \text{ degrees freedom.}$$

Since we want to find t_α so that $P(-t_\alpha < T < t_\alpha) = .99$, we look at our chart to find t_α so $P(T < t_\alpha) = .995$.

We take $t_\alpha = 2.861$, and get interval

$$\left(\bar{X} - 2.861 \frac{S}{\sqrt{n}}, \bar{X} + 2.861 \frac{S}{\sqrt{n}}\right) = (10.126, 16.414)$$

as a 99% confidence interval for μ.

b. (*10 pts*) Construct a 95% confidence interval for σ^2.
Hint: You know the distribution of $(n-1)S^2/\sigma^2$.

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1) = \chi^2(19)$$

Find a, b so that $P(\chi^2(19) < b) = .975$; take $b = 32.852$.

Find a so that $P(\chi^2(19) < a) = .025$; take $a = 8.907$.

$$\therefore P\left(8.907 < \frac{(n-1)S^2}{\sigma^2} < 32.852\right) = .95$$

Rearranging we get:

$$\left(\frac{(n-1)S^2}{32.852}, \frac{(n-1)S^2}{8.907}\right) = (13.967, 51.515)$$

is a 95% confidence interval for σ^2.

5. (*20 points*) A sample $X_1, \ldots, X_n$ of points in $(0,1)$ is drawn from some pdf of the form $f(x;\theta) = \theta x^{\theta-1}$. We wish to test

$$H_0 : \theta = 1; \quad vs \quad H_1 : \theta > 1.$$

Design and explain a test to accomplish this goal, and find it's power function.
Hint: There are a *ton* of reasonable answers here. Two that come to mind: Base something on the mean, or on the distribution of the points - say the number of points in the first half of the strip.

Something on mean:

Reject H_0 and accept H_1 if $\bar{X} > .5 + C$.

(approximate) power of this test:

If θ is fixed; $E[X_i] = \frac{\theta}{\theta+1} = \int_0^1 \theta x^{\theta} dx$ $\quad E[X_i^2] = \int_0^1 \theta x^{\theta+1} = \frac{\theta}{\theta+2}$

$$\gamma_C(\theta) = P_\theta(\bar{X} > .5 + C) = P_\theta\left(\frac{\bar{X} - \frac{\theta}{\theta+1}}{\sigma/\sqrt{n}} > \frac{.5 + C - \frac{\theta}{\theta+1}}{\sigma/\sqrt{n}}\right)$$

$$\approx 1 - \Phi\left(\frac{.5 + C - \frac{\theta}{\theta+1}}{\sqrt{\frac{\theta}{\theta+2} - \left(\frac{\theta}{\theta+1}\right)^2}/\sqrt{n}}\right)$$

$Var(X_i) =$

$$\sigma^2 = \frac{\theta}{\theta+2} - \left(\frac{\theta}{\theta+1}\right)^2$$

# of points in first half of strip:

$$P(X_i \leq \tfrac{1}{2}) = \int_0^{1/2} \theta x^{\theta-1} dx = \frac{1}{2^\theta}.$$

Reject H_0 and accept H_1 if # in first half $< \frac{1}{2} - C$.

(approx) power function; $Y =$ # in first half $\quad Y \sim Bin(n,p)$

$$p = \frac{1}{2^\theta}$$

$$\gamma_C(\theta) = P_\theta\left(Y < \left(\tfrac{1}{2} - C\right)\right) = P\left(\frac{Y - \frac{1}{2^\theta}n}{\sqrt{n\frac{1}{2^\theta}\left(1 - \frac{1}{2^\theta}\right)}} < \frac{\frac{1}{2} - C - \frac{1}{2^\theta}n}{\sqrt{n\frac{1}{2^\theta}\left(1 - \frac{1}{2^\theta}\right)}}\right)$$

$$\approx \Phi\left(\frac{\frac{1}{2} - C - \frac{1}{2^\theta}n}{\sqrt{n\frac{1}{2^\theta}\left(1 - \frac{1}{2^\theta}\right)}}\right)$$

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Practice 2 Solns

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1. (*20 points*)

a. (*10 pts*) Let $\bar{X}$ denote the mean of a random sample size n from a distribution that has mean μ and variance $\sigma^2 = 10$. Find n so that the probability is approximately 0.954 that the random interval $(\bar{X} - \frac{1}{2}, \bar{X} + \frac{1}{2})$ contains μ.

$\mathbb{P}(-2 < Z < 2) \approx .954$ (from chart)

$\frac{\bar{X} - \mu}{\sqrt{10}/\sqrt{n}} \approx N(0,1)$

$\left(\bar{X} - 2\frac{\sqrt{10}}{\sqrt{n}}, X + 2\frac{\sqrt{10}}{\sqrt{n}}\right)$

Want: $2\frac{\sqrt{10}}{\sqrt{n}} = \frac{1}{2} \Rightarrow n = 160$

So: is a .954 conf. interval for μ

$\therefore$ want

Whoops

$\boxed{n = 160}$

b. (*10 pts*) Suppose $\bar{X}$ is the mean of a random sample of size 25 from an $N(\mu, \sigma^2)$ distribution, where $S^2 = 10$. Find an exact 0.95 confidence interval for μ in terms of $\bar{X}$.

$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t$ w/ 24 degrees freedom.

Since: $\mathbb{P}(-2.064 \leq T \leq 2.064) = .95$

$\therefore$

$\left(\bar{X} - 2.064\frac{S}{\sqrt{n}}, \bar{X} + 2.064\frac{S}{\sqrt{n}}\right)$

$= \left(X - 2.064\left(\frac{\sqrt{10}}{5}\right), \bar{X} + 2.064\left(\frac{\sqrt{10}}{5}\right)\right)$ is a 95% confidence interval for μ.

2. (*20 points*) Let $Y_1 < Y_2 < \ldots < Y_n$ denote the order statistics of a random sample of size n from a distribution that has $f(x) = \frac{1}{2\theta}$ for $-\theta < x < \theta$, zero elsewhere.

a. (*10 pts*) Find $\mathbb{P}(-1 < Y_1/\theta < c)$ for $-1 < c < 1$.

$$f_{Y_1}(y_1) = n(1-F_X(y_1))^{n-1} f(y_1) \qquad F_X(y_1) = \int_{-\theta}^{y_1} \frac{1}{2\theta} = \frac{y_1}{2\theta} + \frac{1}{2}$$

$$= n\left(\frac{1}{2} - \frac{y_1}{2\theta}\right)^{n-1} \frac{1}{2\theta} \text{ for } -\theta < y < \theta. \qquad 1 - F_X(y_1) = \frac{1}{2} - \frac{y_1}{2\theta}.$$

$$\mathbb{P}(-\theta < Y_1 < c\theta) = \int_{-\theta}^{c\theta} n\left(\frac{1}{2} - \frac{y_1}{2\theta}\right)^{n-1} dy_1 = \int_{1}^{\frac{1}{2}-\frac{c\theta}{2\theta}} -nu^{n-1}\,du = \int_{\frac{1}{2}-\frac{c}{2}}^{1} nu^{n-1}$$

$$= 1 - \left(\frac{1}{2} - \frac{c}{2}\right)^n.$$

b. (*10 pts*) Find a 90% confidence interval for $-\theta$, if $n = 5$, and $Y_1 = -2.3$.

Find c; so that $\mathbb{P}(-1 < Y_1/\theta < c) = .9$

$$1 - \left(\frac{1}{2} - \frac{c}{2}\right)^5 = .9$$

$$\left(\frac{1}{2} - \frac{c}{2}\right) = \left(\frac{1}{10}\right)^{\frac{1}{5}}$$

$$c = 2\left(\frac{1}{2} - \left(\frac{1}{10}\right)^{\frac{1}{5}}\right) \approx -.262.$$

If $-1 < \frac{Y_1}{\theta} < c$, then $-\frac{1}{Y_1} > \frac{1}{\theta} > \frac{c}{\theta Y_1}$

$\Rightarrow \frac{Y_1}{c} > \theta > -Y_1$ so $\left(-Y_1, \frac{Y_1}{c}\right) = (2.3, 8.78)$

is a 90% confidence interval for θ.

3. (*20 points*) Let $Y_1 < Y_2 < Y_3 < Y_4$ be the order statistics of a random sample of size $n = 4$ from a distribution with pdf $f(x;\theta) = \frac{2x}{\theta^2}$. The hypothesis $H_0 : \theta = 1$ is rejected and $H_1 : \theta > 1$ is accepted if the observed $Y_4 \geq c$. $0<x<\theta$

a. (*10 pts*) Find the constant c so that the significance level is $\alpha = 0.25$.

$$f_{Y_4}(y_n) = 2n\frac{x^{2n-1}}{\theta^{2n}} \Rightarrow f_{Y_4}(y_4) = 8\frac{x^7}{\theta^8} \quad \text{for } 0<x<\theta$$

$$\alpha = P_{\theta=1}(Y_4 \geq c) = \int_c^1 8x^7\,dx = 1 - c^8$$

Want: $\alpha = .25 = 1 - c^8$

$$c^8 = .75$$

$$\boxed{c = .965}$$

b. (*10 pts*) Determine the power function of the test.

$$\gamma_c = \mathbb{P}_\theta(Y_4 \geq c) = \int_c^\theta 8\frac{x^7}{\theta^8}\,dx$$

$$= 1 - \frac{c^8}{\theta^8}$$

If $c = .965$;

$$= 1 - \frac{.75}{\theta^8}$$

95

4. (*20 points*) ~~80~~ Emory students are asked how many A's they expect to receive this semester

Number	0	1	2	≥ 3
Freq	20	40	25	10

It is proposed to fit this with a Poisson distribution.

a. (*10 pts*) Compute the corresponding chi-square test. How many degrees of freedom are associated with this test?

θ = Don't know parameter: ~~assume~~ estimate it:

$\theta \approx \bar{X} = \frac{24}{19}$.

$q_i = P(X=k) \doteq \frac{e^{-\frac{24}{19}} \left(\frac{24}{19}\right)^k}{k!}$

$q_0 = .2822$

$q_1 = .357$

q_2 .225

$q_3 = 1 - q_0 - q_1 - q_2$

$= .134$

Chi square computation: $\frac{(20-26.85)^2}{26.85} + \frac{(40-33.92)^2}{33.92} + \frac{(25-21.375)^2}{21.375} + \frac{(12.73-10)^2}{12.73}$

≈ 4.04. There are 2 degrees of freedom: We start with $k-1=3$, but lose one since we approximate θ.

b. (*10 pts*) Does the data result in the rejection of the Poisson model at the $\alpha = 0.05$ significance level?

We would not reject Poisson model, since

$4.04 < \underbrace{5.991}_{\text{from chart}}$.

5. (*20 points*) $X_1, \ldots, X_{10}$ are independent $N(\mu, \sigma^2)$ random variables, where μ and σ are unknown.

a. (*10 pts*) Explain how to construct a 90% confidence interval for σ^2 using the fact that $(n-1)S^2/\sigma^2$ has the $\chi^2(n-1)$ distribution.

We know: $$\underbrace{P(3.325 \le \chi^2(n-1) \le 16.919)}_{\text{from chart.}} = .9$$

$$.9 = P\left(3.325 \le \frac{(n-1)S^2}{\sigma^2} \le 16.919\right)$$

So: $\left(\frac{9S^2}{16.919}, \frac{9S^2}{3.325}\right)$ is a 90% confidence interval for σ^2

b. (*10 pts*) We wish to test $H_0 : \sigma^2 = 2$ versus $H_1 : \sigma^2 \neq 2$ using our interval at the 90% level. Would we accept or reject H_0 given the following data:

2.55	.95	− 5.24	1.70	− 2.17	1.43	− 0.59	2.66	− 2.59	− .94

sorry for not giving you $\bar{X}$, S^2 here: Didn't think of how time consuming it was. Lesson learned :)

$\bar{X} = .21$, $S^2 = 6.564$.

Get 90% confidence interval of:

$(3.4916, 17.767)$. Since 2 is outside this interval; we reject H_0, and accept H_1