Practice Midterm Exam II

Math 362
2/25/10

Name: ______________________________

Read all of the following information before starting the exam:

- READ EACH OF THE PROBLEMS OF THE EXAM CAREFULLY!
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- A single 8 1/2 $\times$ 11 sheet of notes (double sided) is allowed. Calculators are permitted.
- Copies of normal, t-distribution and χ^2 tables are at the back
- Circle or otherwise indicate your final answers.
- Please keep your written answers clear, concise and to the point.
- This test has . problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Turn off cellphones, etc.
- READ EACH OF THE PROBLEMS OF THE EXAM CAREFULLY!
- Good luck!

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1. (*25 points*) We are able to generate a random variable U which is uniform $(0,1)$ and wish to use it to generate random variables with other distributions.

a. (*10 pts*) Find a transformation to perform to U in order to find X with pdf

$$f_X(x) = \frac{1}{\pi(1+x^2)} \qquad -\infty < x < \infty$$

Hint: $\int \frac{1}{1+x^2}dx = \arctan(x)$.

Want: $F_X(x) = \int_{-\infty}^{x} \frac{1}{\pi(1+t^2)}\,dt = \frac{\arctan(t)}{\pi}\Big|_{-\infty}^{x} = \frac{\arctan(x) + \frac{\pi}{2}}{\pi}$

$= \frac{\arctan(x)}{\pi} + \frac{1}{2} \qquad -\infty < x < \infty.$

$\Rightarrow F^{-1}(x) = \tan\left(\pi x - \frac{\pi}{2}\right),$

So we take $X = F^{-1}(U) = \tan\left(\pi U - \frac{\pi}{2}\right).$

b. (*15 pts*) We repeatedly generate Y with an expo(1) distribution, $f_Y(y) = e^{-y}$ for $0 < y < \infty$. We take $X = Y$ the first time $U \le f_Y(Y)$. Find an expression for the CDF of X as the ratio of two (double) integrals.

Note: You need not identify the distribution of X, or compute the integrals. Just write down the integrals.

Want to find

$$F_X(x) = \mathbb{P}(Y \le x \mid U \le f_Y(Y))$$

$$= \frac{\mathbb{P}(Y \le x, U \le f_Y(Y))}{\mathbb{P}(U \le f_Y(Y))} = \frac{\int_{y=0}^{x}\int_{u=0}^{e^{-y}} du\, e^{-y}\,dy}{\int_{y=0}^{\infty}\int_{u=0}^{e^{-y}} du\, e^{-y}\,dy}$$

Actually, this is not hard to solve:

$$= \frac{\int_0^x e^{-2y}\,dy}{\int_0^\infty e^{-2y}\,dy} = \frac{-\frac{1}{2}e^{-2y}\Big|_0^x}{-\frac{1}{2}e^{-2y}\Big|_0^\infty} = \frac{\frac{1}{2} - \frac{1}{2}e^{-2x}}{\frac{1}{2}} = 1 - e^{-2x}.$$

(In other words $Y \sim \text{expo}(2)$.)

2. (*25 points*) $X_1, \ldots, X_n$ are a sample from a $\Gamma(2, \theta)$ distribution.

a. (*15 pts*) Find the mle $\hat{\theta}$ of θ.

$$L(\theta) = \prod f(x_i;\theta) = \left(\frac{1}{2\theta^2}\right)^{n} \prod (X_i)\, e^{-\sum X_i/\theta}.$$

$$\ell(\theta) = -n\log(2) - 2n\log(\theta) - \log \prod X_i - \frac{\sum X_i}{\theta}$$

$$\ell'(\theta) = \frac{-2n}{\theta} + \frac{\sum X_i}{\theta^2}; \quad \ell'(\theta) = 0 \text{ if } \theta = \frac{\bar{X}}{2},$$

$$\text{so } \hat{\theta} = \frac{\bar{X}}{2}.$$

b. (*15 pts*) Find the mle for $\mathrm{Var}(X_i)$.

Note:

$$\mathrm{Var}(X_i) = \overbrace{2\theta^2}^{\alpha\,\beta^2}, \text{ which is a function of } \theta.$$

Thus the mle for $\mathrm{Var}(X_i)$ is $2\hat{\theta}^2 = 2\left(\frac{\bar{X}}{2}\right)^2$.

$$= \frac{\bar{X}^2}{2}.$$

3. (*20 points*) $X_1, \ldots, X_n$ are drawn from the geometric distribution with parameter θ. That is

$$p(X_i = x) = (1-\theta)^{x-1}\theta, \qquad x = 1, 2, \ldots.$$

Recall, this is a discrete distribution, $\mathbb{E}[\mathrm{X}] = \frac{1}{\theta}$, and variance $\mathrm{Var}(X) = \frac{1-\theta}{\theta^2}$.

a. (*15 pts*) Find the Fisher information $I(\theta)$.

$$f(x;\theta) = (1-\theta)^{x-1}\theta$$

$$\log f(x;\theta) = (x-1)\log(1-\theta) + \log(\theta)$$

$$\frac{\partial \log f(x;\theta)}{\partial \theta} = -\frac{x-1}{1-\theta} + \frac{1}{\theta}$$

$$\frac{\partial^2 \log f(x;\theta)}{\partial^2 \theta} = -\frac{x-1}{(1-\theta)^2} - \frac{1}{\theta^2}$$

Since $\mathbb{E}[X] = \frac{1}{\theta}$,

$$-\mathbb{E}\left[\frac{\partial^2 \log f(x;\theta)}{\partial^2 \theta}\right] = \mathbb{E}\left[\frac{X-1}{(1-\theta)^2} + \frac{1}{\theta^2}\right]$$

$$= \frac{\frac{1}{\theta} - 1}{(1-\theta)^2} + \frac{1}{\theta^2}$$

$$= \frac{1-\theta}{\theta(1-\theta)^2} + \frac{1}{\theta^2}$$

$$= +\frac{1}{\theta(1-\theta)} + \frac{1}{\theta^2}$$

$$= \frac{\cancel{\theta^2} + \theta - \cancel{\theta^2}}{\theta^2(1-\theta)} = \boxed{\frac{1}{\theta(1-\theta)}}.$$

4. (*30 points*) $X_1, \ldots, X_n$ are a random sample from a distribution with pmf $p(x;\theta) = \theta^x(1-\theta)^{1-x}$, x = 0,1. That is, X_i are 0, 1 random variables, and are 1 with probability θ. We wish to test $H_0 : \theta = 1/2$ versus $H_1 : \theta \neq 1/2$.

a. (*15 pts*) Find $-2\log(\Lambda)$.

$$L(\theta) = \theta^{\sum X_i}(1-\theta)^{n-\sum X_i}$$

$$\hat{\theta} = \bar{X}, \text{ so } L(\hat{\theta}) = \bar{X}^{\sum X_i}(1-\bar{X})^{n-\sum X_i}$$

$$L(\theta_0) = \theta_0^{\sum X_i}(1-\theta_0)^{n-\sum X_i}$$

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \left(\frac{\theta_0}{\bar{X}}\right)^{\sum X_i}\left(\frac{1-\theta_0}{1-\bar{X}}\right)^{n-\sum X_i}$$

$$-2\log\Lambda = \sum X_i(\log\theta_0 - \log\bar{X}) + (n-\sum X_i)(\log(1-\theta_0) - \log(1-\bar{X}))$$

b. (*15 pts*) Determine the Wald-type test. (In other words, find χ^2_W)

$$\chi^2_W = \left\{\sqrt{nI(\hat{\theta})}(\hat{\theta} - \theta_0)\right\}^2$$

We found $I(\theta)$ in class for this distribution, but I will repeat:

$$\log f(x;\theta) = x\log(\theta) + (1-x)\log(1-\theta)$$

$$\frac{\partial \log f(x;\theta)}{\partial\theta} = \frac{x}{\theta} - \frac{1-x}{1-\theta}$$

$$\frac{\partial^2 \log f(x;\theta)}{\partial^2\theta} = -\frac{x}{\theta^2} - \frac{1-x}{(1-\theta)^2};$$

Since $E[X] = \theta$

$$I(\theta) = -E\left[\frac{-x}{\theta^2} - \frac{1-x}{(1-\theta)^2}\right] = \frac{1}{\theta} + \frac{1}{1-\theta} = \frac{1-\theta+\theta}{\theta(1-\theta)} = \frac{\theta}{1-\theta}.$$

Thus; as $\hat{\theta} = \bar{X}$

$$\chi^2_W = \left\{\sqrt{n\frac{\bar{X}}{1-\bar{X}}}(\bar{X} - \theta_0)\right\}^2.$$