Math 362, Problem set 2

Due 2/9/10

1. (5.2.17) Let $Y_1 < Y - 2 < Y_3 < Y_4$ be the order statistics of a random sample of size $n = 4$ from a distribution with pdf $f(x) = 2x$, $0 < x < 1$, zero elsewhere.

   (a) Find the joint pdf of Y_3 and Y_4.

   (b) Find the conditional pdf of Y_3, given $Y_4 = y_4$.

   (c) Evaluate $\mathbb{E}[Y_3|y_4]$.

   Answer: Recall:

   $$f_{Y_1,Y_2,Y_3,Y_4}(y_1, y_2, y_3, y_4) = 4!(2y_1)(2y_2)(2y_3)(2y_4)$$

   for $0 < y_1 < y_2 < y_3 < y_4 < 1$. We have:

   $$\begin{aligned} f_{Y_3,Y_4}(y_3, y_4) &= \int_{y_2=0}^{y_3} \int_{y_1=0}^{y_2} f_{Y_1,Y_2,Y_3,Y_4}(y_1, y_2, y_3, y_4) \\ &= 48y_3^5 y_4 \end{aligned}$$

   for $0 < y_3 < y_4 < 1$.

   For (b):

   $$f_{Y_3|Y_4}(y_3|y_4) = \frac{f_{Y_3,Y_4}(y_3, y_4)}{f_{Y_4}(y_4)} = \frac{6y_3^5}{y_4^6}$$

   for $0 < y_3 < y_4$.

   For (c), we have that

   $$\mathbb{E}[Y_3|Y_4 = y_4] = \int_0^{y_4} \frac{6y_3^6}{y_4^6} = \frac{6}{7}y_4.$$

2. (5.4.9) Let $\bar{X}$ denote the mean of a random sample of size 25 from a gamma-type distribution with $\alpha = 4$ and $\beta > 0$. Use the Central Limit theorem to find an approximate 0.954 confidence interval for μ, the mean of the gamma distribution.

   Hint: Use the random variable $(\bar{X} - 4\beta)/\sqrt{4\beta^2/25}$

Answer: We note that $\mathbb{P}(-2 < Z < 2) = 0.954$. Using the fact that $(\bar{X} - 4\beta)/\sqrt{4\beta^2/25}$ is approximately $N(0,1)$, we know that

$$\begin{aligned}
.954 &= \mathbb{P}(-2 < (\bar{X} - 4\beta)/\sqrt{4\beta^2/25} < 2) \\
&= \mathbb{P}(-2 < 5\bar{X}/2\beta - 10 < 2) \\
&= \mathbb{P}(8 < 5\bar{X}/2\beta < 12) \\
&= \mathbb{P}(\frac{5\bar{X}}{6} < 4\beta < \frac{5\bar{X}}{4}).
\end{aligned}$$

This gives a confidence interval for $\mu = 4\beta$ of $(\frac{5\bar{x}}{6}, \frac{5\bar{x}4}{)}$. Note that the book gives a confidence interval for β instead of 4β.

3. (5.4.13) Let $Y_1 < Y_2 < \cdots < Y_n$ denote hte order statistics of a random sample of size n from a distribution that has pdf $f(x) = 3x^2/\theta^3$, $0 < x < \theta$, zero elsewhere.

   (a) Show that $\mathbb{P}(c < Y_n/\theta < 1) = 1 - c^{3n}$ where $0 < c < 1$.h

   (b) If n is 4 and if the observed value of Y_4 is 2.3, what is a 95% confidence interval for θ.

   Answer:

   We know that

   $$f_{Y_n}(y_n) = 3n(\frac{x^3}{\theta^3})^{n-1}\frac{x^2}{\theta^3}.$$

   and

   $$\mathbb{P}(c\theta < Y_n < \theta) = \int_{c\theta}^{\theta} 3n\frac{x^{3n-1}}{\theta^{3n}}dx = 1 - c^{3n}.$$

   For (b), we note that

   $$\mathbb{P}(c\theta < Y_4 < \theta) = 1 - c^{12} = .95,$$

   if $c = (.5)^{1/12}$. That is,

   $$.95 = \mathbb{P}((.5)^{1/12}\theta < Y_4 < \theta) = \mathbb{P}(Y_4 < \theta < \frac{Y_4}{(.5)^{1/12}})$$

   This gives a 95% confidence interval for θ of $(2.3, 2.3/(.5)^{1/12})$.

4. (5.4.16) When 100 tacks were thrown on a table, 60 of them landed point up. Obtain a 95% confidence interval for the probability that a tack of this type will land point up. Assume independence.

   Answer: We have that $\bar{p} = .6$, and have a 95% (approximate) confidence interval of

   $$(\bar{p} - 1.96\sqrt{\bar{p}(1-\bar{p})/n}, \bar{p} + 1.96\sqrt{\bar{p}(1-\bar{p})/n}) = (.503, .696)$$

5. (5.4.14) Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$ where both parameters μ and σ^2 are unknown. A confidence interval for σ^2 can be found as follows: We know $(n-1)S^2/\sigma^2$ is a random variable with a $\chi^2(n-1)$ distribution. Thus we can find constants a and b so that $\mathbb{P}((n-1)S^2/\sigma^2 < b) = 0.975$ and $\mathbb{P}(a < (n-1)S^2/\sigma^2 < b) = 0.95$.

   (a) Show that this second probability statement can be written as

   $$\mathbb{P}((n-1)S^2/b < \sigma^2 < (n-1)S^2/a) = 0.95$$

   (b) If $n = 9$ and $s^2 = 7.93$ (here s^2 is the actual value of S^2 given data), find a 95% confidence interval for σ^2.

   (c) If μ is known, how would you modify the preceding procedure for finding a confidence interval for σ^2.

   Answer: For (a), just rearrange $\mathbb{P}(a < (n-1)S^2/\sigma^2 < b)$; note the inequalities flip when we take the reciprocal.

   For $n = 9$, we find (from Table II) that $b = 17.535$ and $a = 2.18$. Thus we have a confidence interval of

   $$(3.618, 29.1).$$

   For (c), note that if we know μ we know that $(\bar{X} - \mu)/(\sigma/\sqrt{n}) \sim N(0, 1)$, so $(\bar{X} - \mu)^2/\sigma^2/\sqrt{n}$ has a $\chi^2(1)$ distribution. We can use this as above to find a confidence interval for σ.

6. (3.6.2) Let T have a t-distribution with 14 degrees of freedom. Determine b so that $\mathbb{P}(-b < T < b) = 0.9$. Use Table IV.

   Answer:

   Take $b = 1.761$.