# Math 362, Problem set 3

Due 2/16/10

1. (5.4.18) Using the assumptions behind the confidence interval given in expression (5.4.17), show that

$$\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}\Big/\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}} \to_P 1.$$

   *Answer:*

   The real key here is to address why the assumption $n_1/n \to \lambda_1$ and $n_2/n \to \lambda_2$ is important. We know $S_1^2 \to_p \sigma_1^2$, but we cannot simply cannot say:

$$\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}} \to_p \sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}$$

   because we are taking this limit *as* $n_1, n_2 \to \infty$ so they don't make sense in the second statement. So let's do an old trick from calculus:

$$\frac{\sqrt{n}}{\sqrt{n}}\frac{\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}} = \frac{\sqrt{\frac{nS_1^2}{n_1}+\frac{nS_2^2}{n_2}}}{\sqrt{\frac{n\sigma_1^2}{n_1}+\frac{n\sigma_2^2}{n_2}}}.$$

   Now the limit as $n \to \infty$ makes sense on the top and bottom, that is:

$$\frac{\sqrt{\frac{nS_1^2}{n_1}+\frac{nS_2^2}{n_2}}}{\sqrt{\frac{n\sigma_1^2}{n_1}+\frac{n\sigma_2^2}{n_2}}} \to_p \frac{\sqrt{\lambda_1\sigma_1^2+\lambda_2\sigma_2^2}}{\sqrt{\lambda_1\sigma_1^2+\lambda_2\sigma_2^2}} = 1$$

2. (5.4.24) Let $\bar{X}$ and $\bar{Y}$ be the means of two independent random samples, each of size $n$, from the respective distributions $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$ where the common variance $\sigma^2$ is known. (Note: there is a type in the book, writing $\sigma_2$ instead of $\sigma^2$, and my statement is correct). Find $n$ such that

$$\mathbb{P}(\bar{X}-\bar{Y}-\sigma/5 < \mu_1-\mu_2 < \bar{X}-\bar{Y}+\sigma/5) = 0.9.$$

*Answer:* Note that $X_i - Y_i \sim N(\mu_1 - \mu_2, 2\sigma^2)$ so

$$\mathbb{P}(-1.645 \leq \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{2\sigma^2/n}} \leq 1.645) = .9$$

Rearranging,

$$.9 = \mathbb{P}(\bar{X} - \bar{Y} - 1.645\sqrt{2}\sigma/\sqrt{n} < \mu_1 - \mu_2 < \bar{X} - \bar{Y} + 1.645\sqrt{2}\sigma/\sqrt{n}).$$

Now, simply choose $n$ so that $1.645\sqrt{2}/\sqrt{n} < 1/5$, in other words take $n > 50(1.645)^2$, so that $n = 136$ suffices.

3. (5.5.1) Show that the approximate power function given in expression (5.5.12) of Example 5.5.3 is a strictly increasing function of $\mu$. Show, then, that the test discussed in this example has approximate size $\alpha$ for testing

$$H_0 : \mu \leq \mu_0 \text{ versus } H_1 : \mu > \mu_0.$$

*Answer:* We have approximate power function

$$\gamma_C(\mu) = \Phi(-z_\alpha - \frac{\sqrt{n}}{\sigma}(\mu - \mu_0)),$$

so

$$\begin{aligned}
\frac{d}{d\mu}\Phi(-z_\alpha - \frac{\sqrt{n}}{\sigma}(\mu - \mu_0)) &= \frac{d}{d\mu}\int_{-\infty}^{-z_\alpha - \frac{\sqrt{n}}{\sigma}(\mu-\mu_0)} \frac{1}{\sqrt{2\pi}}e^{-1/2x^2}dx \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(-z_\alpha - \frac{\sqrt{n}}{\sigma}(\mu - \mu_0))^2\right)\frac{\sqrt{n}}{\sigma} > 0,
\end{aligned}$$

as the product of positive numbers. Thus $\gamma_C(\mu)$ is an increasing function of $\mu$.

The approximate size of the discussed test is

$$\alpha = \max_{\mu \leq \mu_0} \gamma_C(\mu) = \gamma_C(\mu_0) = \alpha,$$

where we used the fact that $\gamma_C(\mu)$ is an increasing function of $\mu$ to take the maximum.

4. (5.5.5) Let $X_1, X_2$ be a random sample of size $n = 2$ from the distribution having pdf $f(x;\theta) = (1/\theta)e^{-x/\theta}$ for $0 < x < \infty$, zero elsewhere. That is, $X_i \sim \text{expo}(1/\theta)$ or equivalently $X_i \sim \Gamma(1, \theta)$. We reject $H_0 : \theta = 2$ and accept $H_1 : \theta = 1$ if the observed values of $X_1, X_2$, say $x_1, x_2$, are such that

$$\frac{f(x_1;2)f(x_2;2)}{f(x_1;1)f(x_2;1)} \leq \frac{1}{2}$$

Here $\Omega = \{\theta : \theta = 1, 2\}$. Find the significance level of the test and the power of the test when $H_0$ is false.

*Answer:*

Rearranging, we find the test that

$$\frac{f(X_1;2)f(X_2;2)}{f(X_1;1)f(X_2;1)} \leq \frac{1}{2}$$

is the test

$$\frac{(1/4)\exp(-(X_1+X_2)/2)}{exp(-(X_1+X_2)} \leq \frac{1}{2}.$$

More rearranging, and we get that this is equivalent to:

$$X_1 + X_2 \leq 2\ln(2),$$

and our power function is

$$\gamma_C(\theta) = \mathbb{P}_\theta(X_1 + X_2 \leq 2\ln(2)).$$

Our size is

$$\alpha = \mathbb{P}_{\theta=2}(X_1 + X_2 \leq 2\ln(2)).$$

In this case, $X_1+X_2 \sim \Gamma(2,2)$ as the sum of two $\Gamma(1,2)$ random variables. Thus

$$\alpha = \int_0^{2\ln(2)} \frac{1}{4}xe^{-x/2}dx = \frac{1}{2}(1-\ln(2)) \approx 0.1534.$$

In the case that $\theta = 1$, $X_1 + X_2 \sim \Gamma(2,1)$ so that

$$\gamma_C(1) = \int_0^{2\ln(2)} xe^{-x}dx = \frac{3}{4} - \frac{1}{2}\ln(2) \approx 0.403,$$

which is the power of the test if $H_0$ is false.

5. (5.5.9) Let $X$ have a Poisson distribution with mean $\theta$. Consider the simple hypothesis $H_0 : \theta = \frac{1}{2}$, and the alternative composite hypothesis $H_1 : \theta < \frac{1}{2}$. Thus $\Omega = \{\Theta : 0 < \theta \leq \frac{1}{2}\}$. Let $X_1, \ldots, X_{12}$ denote a random sample of size 12 from this distribution. We reject $H_0$ if and only if the observed value of $Y = X_1 + \cdots + X_{12} \leq 2$. If $\gamma(\theta)$ is the power function of the test, find the powers $\gamma(1/2)$, $\gamma(1/3)$, $\gamma(1/4)$, $\gamma(1/6)$ and $\gamma(1/12)$. Sketch the graph of $\gamma(\theta)$. What is the significance level of this test?

   *Answer:*

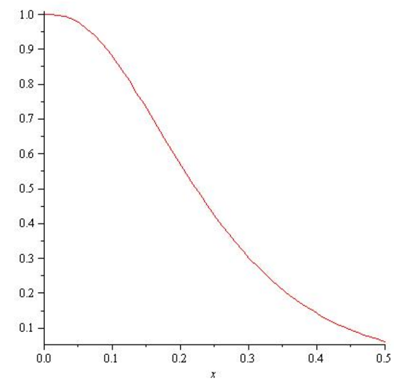
   As the sum of 12 Poisson($\theta$) random variables, $Y \sim Poisson(12\theta)$. Thus

$$\gamma(\theta) = \mathbb{P}_\theta(Y \leq 2) = e^{-12\theta} + e^{-12\theta}(12\theta) + \frac{e^{-12\theta}(12\theta)^2}{2}.$$

   We can compute

$$\begin{aligned}
\gamma(1/2) &\approx 0.06196880442 \\
\gamma(1/3) &\approx 0.2381033056 \\
\gamma(1/4) &\approx 0.4231900811 \\
\gamma(1/6) &\approx 0.6766764160 \\
\gamma(1/12) &\approx 0.9196986030
\end{aligned}$$

A plot




The significance level is the same as the size is $\alpha = \gamma(1/2) \approx 0.06196880442$.

6. (5.5.11) Let $Y_1 < Y_2 < Y_3 < Y_4$ be the order statistics of a random sample of size $n = 4$ from a distribution with pdf $f(x;\theta) = 1/\theta$, $0 < x < \theta$, zero elsewhere, where $0 < \theta$. The hypothesis $H_0 = 1$ is rejected and $H_1 : \theta > 1$ is accepted if the observed $Y_4 \geq c$.

   (a) Find the constant $c$ so that the significance level is $\alpha = 0.05$.

   (b) Determine the power function of the test.

   *Answer:*

   Recall
$$f_{Y_4}(y_4) = 4\big(\frac{y}{\theta}\big)^3 \frac{1}{\theta},$$
   so
$$\gamma_C(\theta) = \mathbb{P}_\theta(Y_4 \geq c) = \int_c^\theta f_{Y_4}(y_4) dy_4 = 1 - \big(\frac{c}{\theta}\big)^4.$$
   In order to find $c$ so that the significance level is $\alpha = 0.05$, we want to find $c$ so that
$$0.05 = \alpha = \gamma_C(\theta) = 1 - c^4.$$
   In other words, we take $c = (.95)^{1/4} \approx 0.9872585449$.

   Then the power function is
$$\gamma_C(\theta) = 1 - \big(\frac{c}{\theta}\big)^4. = 1 - \frac{.95}{\theta^4}.$$