Math 362, Problem Set 6

Due 3/23/11

1. (6.1.2) Let $X_1, X_2, \ldots, X_n$ be a random sample from a $\Gamma(\alpha = 3, \beta = \theta)$ distribution, $0 < \theta < \infty$. Determine the mle of θ.

   Answer

   We have

$$L(\theta) = \frac{1}{\theta^{3n}2^n} \prod (X_i^2) e^{-\sum X_i/\theta}$$
$$\ell(\theta) = -3n\log(\theta) - n\log(2) + \sum 2\log(X_i) - \sum X_i/\theta$$
$$\ell'(\theta) = \frac{-3n}{\theta} + \frac{\sum X_i}{\theta^2}.$$

   Solving $\ell'(\theta) = 0$, we have
$$\hat{\theta} = \frac{1}{3}\bar{X}.$$

2. (6.1.5) Suppose $X_1, \ldots, X_n$ are iid with pdf $f(x;\theta) = 2x/\theta^2$, $0 < x \leq \theta$, zero elsewhere. Find

   (a) The mle $\hat{\theta}$ for θ.

   (b) The constant c so that $\mathbb{E}[c\hat{\theta}] = \theta$.

   (c) The mle for the median of the distribution.

   Answer:

   We have that
$$L(\theta) = \begin{cases} \frac{2^n \prod(X_i)}{\theta^{2n}} & \max X_i < \theta \\ 0 & \max(X_i) > \theta. \end{cases}$$

   Since $\frac{2^n \prod(X_i)}{\theta^{2n}}$ is a decreasing function of θ, $L(\theta)$ is maximized when $\theta = \max(X_i)$. That is $\hat{\theta} = \max(X_i) = Y_n$.

   Using the pdf of the order statistics, and the fact that $F_X(x) = n\frac{x^2}{\theta^2}$.
$$\mathbb{E}[\hat{\theta}] = \int_0^\theta 2n\frac{x^{2n}}{\theta^{2n}} = \frac{2n}{2n+1}\theta.$$

Thus we take $c = \frac{2n+1}{2n}$.

For (b), we compute that the median of X's distribution is Q_2 so that

$$\frac{1}{2} = \int_0^{Q_2} \frac{2x}{\theta^2} = \frac{Q_2^2}{\theta^2}.$$

Thus $Q_2 = \frac{\theta}{\sqrt{2}}$.

Thus $\hat{Q}_2 = \frac{Y_n}{\sqrt{2}}$.

3. (6.1.9) Suppose $X_1, \dots, X_n$ are iid with pdf $f(x;\theta) = (1/\theta)e^{-x/\theta}$. Find the mle of $\mathbb{P}(X \leq 2)$.

   Answer:

   For this distribution we have that

$$\ell(\theta) = -n\log(\theta) - \sum X_i/\theta$$
$$\ell'(\theta) = \frac{-n}{\theta} + \frac{\sum X_i}{\theta^2}.$$

   Solving $\ell'(\theta) = 0$, we see that $\hat{\theta} = \bar{X}$.

   Since

$$\mathbb{P}(X \leq 2) = \int_0^2 (1/\theta)e^{-x/\theta} = 1 - e^{-2/\theta}.$$

   Thus the mle of $\mathbb{P}(X \leq 2)$ is

$$1 - e^{-2/\bar{X}}.$$

4. (6.1.10) If $X_1, X_2, \dots X_n$ be a random sample from a Bernoulli distribution with parameter p. If p is restricted so that we know that $\frac{1}{2} \leq p \leq 1$, find the mle of this parameter.

   Answer:

   We have that

$$L(\theta) = p^{\sum X_i}(1-p)^{n-\sum X_i}$$
$$\ell(\theta) = \sum(X_i)\log(p) + (n - \sum X_i)\log(1-p)$$
$$\ell'(\theta) = \frac{\sum X_i}{p} + \frac{n - \sum X_i}{1-p}.$$

   Setting $\ell'(\theta) = 0$, and solving we find $\theta = \bar{X}$. On the other hand since $\ell'(\theta) > 0$ for $\theta < \bar{X}$ and $\ell'(\theta) < 0$ for $\theta < \bar{X}$. We see that if $\bar{X} < \frac{1}{2}$, we have $\ell(\theta)$ is maximized at $\theta = \frac{1}{2}$. Thus $\hat{\theta} = \max\{\frac{1}{2}, \bar{X}\}$.

5. (6.2.1) Prove that $\bar{X}$, the mean of a random sample of size n from a distribution that is $N(\theta, \sigma^2)$ is, for every known $\sigma^2 > 0$, an efficient estimator of θ.

   Answer:

   We compute

$$
\begin{aligned}
f(x;\theta) &= \frac{1}{\sqrt{2\pi\sigma^2}} exp\big(- \frac{1}{2\sigma^2}(x-\theta)^2\big) \\
\log(f(x;\theta)) &= -\frac{1}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x-\theta)^2 \\
\frac{\partial \log(f(x;\theta))}{\partial\theta} &= -\frac{x-\theta}{\sigma^2} \\
\frac{\partial^2 \log(f(x;\theta))}{\partial\theta} &= -\frac{1}{\sigma^2}.
\end{aligned}
$$

   Thus

$$
I(\theta) = -\mathbb{E}[-\frac{1}{\sigma^2}] = \frac{1}{\sigma^2}.
$$

   On the other hand, $\text{Var}(\bar{X}) = \frac{\sigma^2}{n}$. We have that $k(\theta) = \mathbb{E}[\bar{X}] = \theta$, so that $k'(\theta) = 1$. Thus the Rao-Cramer bound is $\frac{\sigma^2}{n}$ which matches the variance of $\bar{X}$. Thus $\bar{X}$ is an efficient estimator of θ.

6. (6.2.7') Let X have a gamma distribution with $\alpha = 3$ and $\beta = \theta > 0$.

   (a) Find the Fisher information $I(\theta)$.

   (b) If $X_1, \ldots, X_n$ is a random sample from this distribution, show that the mle of θ is an efficient estimator of θ.

   (c) What is the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$?

   Note: I changed $\alpha = 4$ in the original problem to $\alpha = 3$ since you computed the mle for θ in this case above.

   Answer:

   We have that

$$
\begin{aligned}
f(x;\theta) &= \frac{1}{2\theta^3}x^2 e^{-x/\theta} \\
\log(f(x;\theta)) &= -\log(2) - 3\log(\theta) + 2\log(x) - \frac{x}{\theta} \\
\frac{\partial \log(f(x;\theta))}{\partial\theta} &= \frac{-3}{\theta} + \frac{x}{\theta^2} \\
\frac{\partial^2 \log(f(x;\theta))}{\partial\theta} &= \frac{3}{\theta^2} - \frac{2x}{\theta^3}.
\end{aligned}
$$

Since $\mathbb{E}[X] = 3\theta$, we see that $I(\theta) = \frac{3}{\theta^2}$.

We found the mle for θ in problem one to be $\bar{X}/3$. Note that $\text{Var}(\hat{\theta}) = \frac{1}{9n}\text{Var}(X_i) = \frac{\theta^2}{3n}$. The Rao-Cramer bound is $\frac{\theta^2}{3n}$ as well, so $\hat{\theta}$ is efficient.

Note that $\sqrt{n}(\hat{\theta} - \theta) \to N(0, \frac{1}{I(\theta)})$.