Math 362, Problem set 7

Due 3/30/10

1. (6.1.11) Let $X_1, \ldots, X_n$ be a random sample from an $N(\theta, \sigma^2)$ distribution, where σ^2 is fixed and known, and $-\infty < \theta < \infty$.

   (a) Show that the mle of θ is $\bar{X}$

   (b) If θ is restricted by $0 \leq \theta < \infty$, show that the mle of θ is $\hat{\theta} = \max\{0, \bar{X}\}$.

   Answer: We have that

   $$L(\theta) = \left(\frac{1}{\sqrt{2\pi\sigma}}\right)^n \exp\left(-\frac{1}{2\sigma}\sum(X_i - \theta)^2\right)$$

   so

   $$\ell(\theta) = -\frac{n}{2}\log(2\pi\sigma) - \frac{1}{2\sigma^2}\sum(X_i - \theta)^2$$

   and

   $$\ell'(\theta) = -\frac{1}{\sigma^2}\sum(X_i - \theta)$$

   Setting $\ell'(theta) = 0$, we see that $\hat{\theta} = \bar{X}$.

   For (b), we observe that $\ell'(\theta) < 0$ for $\theta < \bar{X}$ and $\ell'(\theta) < 0$ for $\theta > \bar{X}$. Thus if we know that $\theta \geq 0$, and $\bar{X} < 0$, $\ell(\theta)$ is maximized at 0. That is $\hat{\theta} = \max\{0, \bar{X}\}$

2. Let $X_1, \ldots X_n$ be a random sample from an $N(0, \theta)$ distribution. We want to estimate the standard deviation $\sqrt{\theta}$. Find the constant c so that $Y = c\sum|X_i|$ is an unbiased estimator of $\sqrt{\theta}$ and determine it's efficiency.

   We note that

   $$\begin{aligned}\mathbb{E}[|X_i|] &= 2\int_0^\infty \frac{1}{\sqrt{2\pi\theta}} x e^{-x^2/2\theta} dx \\ &= 2\int_0^\infty \frac{\sqrt{\theta}}{\sqrt{2\pi}} e^{-u} du = \sqrt{2\theta/\pi}.\end{aligned}$$

   Thus we take $c = \frac{1}{n}\sqrt{\pi/2}$.

We compute $I(\theta)$ next. We have

$$\begin{aligned}
f(x;\theta) &= \frac{1}{\sqrt{2\pi\theta}} exp\big(-\frac{1}{2\theta}x^2\big) \\
\log(f(x;\theta)) &= -\frac{1}{2}\log(2\pi\theta) - \frac{1}{2\theta}x^2 \\
\frac{\partial \log(f(x;\theta))}{\partial\theta} &= -\frac{1}{2\theta} + \frac{x^2}{2\theta^2} \\
\frac{\partial^2 \log(f(x;\theta))}{\partial\theta} &= \frac{1}{2\theta^2} - \frac{x^2}{\theta^3}.
\end{aligned}$$

Thus

$$I(\theta) = \mathbb{E}\big[-\frac{\partial^2 \log(f(X;\theta))}{\partial\theta}\big] = \mathbb{E}\big[\frac{X^2}{\theta^3} - \frac{1}{2\theta^2}\big] = \frac{1}{2\theta^2},$$

as $\mathbb{E}[X^2] = \theta$.

We have $k(\theta) = \sqrt{\theta}$, and $k'(\theta) = \frac{1}{2\sqrt{\theta}}$. Thus the Rao-Cramer lower bound is

$$\frac{(k'(\theta))^2}{nI(\theta)} = \frac{\theta}{2n}.$$

On the other hand,

$$\mathrm{Var}(Y) = c^2 n \mathrm{Var}(|X_i|) = c^2 n(\mathbb{E}[X_i^2] - \mathbb{E}[|X_i|]^2) = \theta n \frac{(\pi-2)}{2}.$$

Thus we see that the efficiency is

$$\frac{(k'(\theta))^2}{\mathrm{Var}(Y) n I(\theta)} = \frac{1}{\pi - 2}.$$

3. (6.2.14) Let S^2 be the sample variance of a random sample of size $n > 1$ from $N(\mu, \theta)$, $0 < \theta < \infty$, where μ is known. We know $\mathbb{E}[S^2] = \theta$.

   (a) What is the efficiency of S^2?

   (b) Under these conditions, what is the mle $\hat{\theta}$ of θ?

   (c) What is the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$.

   Answer:

$$\begin{aligned}
f(x;\theta) &= \frac{1}{\sqrt{2\pi\theta}} exp\big(-\frac{1}{2\theta}(x-\mu)^2\big) \\
\log(f(x;\theta)) &= -\frac{1}{2}\log(2\pi\theta) - \frac{1}{2\theta}(x-\mu)^2 \\
\frac{\partial \log(f(x;\theta))}{\partial\theta} &= -\frac{1}{2\theta} + \frac{(x-\mu)^2}{2\theta^2} \\
\frac{\partial^2 \log(f(x;\theta))}{\partial\theta} &= \frac{1}{2\theta^2} - \frac{(x-\mu)^2}{\theta^3}.
\end{aligned}$$

and as before $I(\theta) = \frac{1}{2\theta^2}$, and $k(\theta) = \theta$, so $k'(\theta) = 1$. On the other hand, since S^2 has a $\chi^2(n-1)$ distribution, we know that $\mathrm{Var}(S^2) = \frac{\theta^2}{(n-1)^2}\mathrm{Var}((n-1)S^2/\theta) = \frac{2\theta^2}{n-1}$. We have that the efficiency of S^2 is

$$\frac{2\theta^2}{n\mathrm{Var}(S^2)} = \frac{n-1}{n}.$$

We have

$$\begin{aligned}\ell(\theta) &= -\frac{n}{2}\log(2\pi\theta) - \frac{1}{2\theta}\sum(X_i-\mu)^2 \\ \ell'(\theta) &= -\frac{n}{2\theta} + \frac{1}{2\theta^2}\sum(X_i-\mu)^2.\end{aligned}$$

Solving, we see $\hat{\theta} = \frac{1}{n}\sum(X_i-\mu)^2$.

For (c), we see that $\sqrt{n}(\hat{\theta}-\theta) \to N(0, 1/I(\theta)) = N(0, 2\theta^2)$.

4. (6.3.5) Let $X_1, \ldots, X_n$ be a random sample from a $N(\mu_0, \theta)$ distribution, where $0 < \theta < \infty$ and μ_0 is known. Show that the likelihood ratio test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ can be based upon the statistic $W = \sum_{i=1}^n (X_i - \mu_0)^2/\theta_0$. Determine the null distribution of W (that is, the distribution of W given that $\theta = \theta_0$, and give, explicitly a rejection rule for a level α test.

   Hint/Note: If $\theta = \theta_0$, so that $X_i \sim N(\mu_0, \theta)$ what is the distribution of $(X_i - \mu_0)^2/\theta_0$? It's one we know. Maybe figure out the distribution of $(X_i - \mu_0)/\sqrt{\theta_0}$ first.

   Answer:

   We have

$$L(\theta) = \left(\frac{1}{\sqrt{2\pi\theta}}\right)^{n/2} \exp\big(-\frac{1}{2\theta}\sum(x_i-\mu)^2.$$

   Thus

$$L(\theta_0) = \left(\frac{1}{\sqrt{2\pi\theta_0}}\right)^{n/2} e^{-W/2}.$$

   We found $\hat{\theta}$ in the last problem, so

$$\text{Ł}(\hat{\theta}) = \left(\frac{1}{\sqrt{2\pi\frac{1}{n}\sum(x_i-\mu)^2}}\right)^{n/2} e^{-n/2}$$

   Combining, we have

$$\Lambda = n^{-n/2}e^{n/2}W^{n/2}e^{-W/2},$$

   so this test depends on W as desired. Note that $\Lambda \leq c$ is the same as $W \leq c_1$ or $W \geq c_2$ for some constants c_1 and c_2. Since $W \sim \chi^2(n)$, we take $c_1 = \chi^2_{\alpha/2}(n)$ and $c_2 = \chi^2_{1-\alpha/2}(n)$ to get a test of size α.

5. (6.3.8) Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with mean $\theta > 0$.

   (a) Show that the likelihood ratio test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ is based upon the statistic $Y = \sum_{i=1}^n X_i$. Obtain the null distribution of Y.

   (b) For $\theta_0 = 2$ and $n = 5$, find the significance level of the test that rejects H_0 if $Y \leq 4$, or $Y \geq 17$.

   Note: For (a), show that the test is of the form reject H_0 if $f(Y) > c$. It will not immediately look like it is of the form $Y > c$. The null distribution of Y is the distribution of Y if the null hypothesis is true.

   Answer: We have

$$L(\theta) = e^{-n\theta} \frac{\theta^{\sum X_i}}{\prod X_i}.$$

   We also know $\hat{\theta} = \bar{X}$, so

$$\Lambda = e^{-n\theta_0} \frac{(\theta_0)^Y e^Y}{(Y/n)^Y}$$

   This is a function of Y; which is $Poisson(n\theta)$.

   For (b), we have that $Y \sim Poisson(10)$, and hence the size of this test is $(.029) + (1 - .973)$.