Math 362, Problem set 8

Due 4/13/10 (okay to turn in on 4/15)

1. (6.3.18)

   Answer: We have that the mle $\hat{\theta} = Y_n$. We have

   $$L(\theta) = \frac{1}{\theta^n}.$$

   if $Y_n \leq \theta$.

   Computing

   $$\Lambda = \frac{L(\theta_0)}{L(Y_n)} = \begin{cases} (\frac{Y_n}{\theta_0})^n & \text{if } Y_n \leq \theta_0 \\ 0 & \text{if } Y_n > \theta_0. \end{cases}$$

   For (b), if $Z = -2\log\Lambda = -2n\log(Y_n/\theta_0)$, then $Y_n = \theta_0 e^{-Z/2n}$. Thus if $Z = g(Y_n)$, $g^{-1}(x) = \theta_0 e^{-x/2n}$ and hence $|(g^{-1})'(x)| = \frac{\theta_0}{2n}e^{-x/2n}$. We also know that $f_{Y_n}(y) = n\frac{y^{n-1}}{\theta_0^n}$, for $0 < y < 1$.

   Thus

   $$\begin{aligned} f_Z(z) &= f(g^{-1}(z)) * (g^{-1})'(z) \\ &= n\frac{g^{-1}(z)^{n-1}}{\theta_0^n} \cdot \frac{\theta_0}{2n}e^{-z/2n} \\ &= \frac{1}{2}e^{-z/2} \end{aligned}$$

   for $0 < z < \infty$. Thus $Z \sim \Gamma(1,2) = \chi^2(2)$ as desired.

2. (6.4.3)

   Answer:

   We have

   $$L(\theta_1, \theta_2) = (1/\theta_2)^n e^{-\sum(x_i - \theta_1)/\theta_2}.$$

   so long as $\min X_i \leq \theta_1$. This is a decreasing function of θ_1, so to maximize L with respect to θ_1, we take $\hat{\theta_1} = \min X_i$.

   We have

   $$\ell(\theta_1, \theta_2) = -n\log(\theta_2) - \frac{\sum(x_i - \theta_1)}{\theta_2}.$$

Thus
$$\frac{\partial \ell(\theta_1, \theta_2)}{\partial \theta_2} = \frac{-n}{\theta_2} + \frac{\sum (x_i - \theta_1)}{\theta_2^2}.$$

Setting this to be zero we see that this is minimized when $\theta_2 = \frac{\sum (x_i - \theta_1)}{n}$. Thus we get the mle $\hat{\theta_2} = \frac{\sum (x_i - \min(x_i))}{n}$.

3. (7.1.4)

   Answer: We have
$$\mathbb{E}[k_1 X_1 + k_2 X_2] = k_1 \mathbb{E}[X_1] + k_2 \mathbb{E}[X_2] = (k_1 + k_2)\theta$$
   so this is unbiased so long as $k_1 + k_2 = 1$.

   We have that
$$\mathrm{Var}(k_1 X_1 + k_2 X_2) = k_1^2 \mathrm{Var}(X_1) + k_2^2 \mathrm{Var}(X_2) = (2k_1^2 + k_2^2)\mathrm{Var}(X_2)$$
   Thus we wish to minimize $2k_1^2 + k_2^2$ subject to $k_1 + k_2 = 1$. We could do this with Lagrange multipliers, but it is easier to see that if $k_1 = x$, $k_2 = 1 - x$ so that we wish to minimize
$$2x^2 + (1 - x)^2 = 3x^2 - 2x + 1$$
   which is minimized when $x = k_1 = 1/3$ and $1 - x = k_2 = 2/3$.

4. (7.1.5)

   Answer:

   We have that $X_1, \ldots, X_{25}$ are $N(\theta, 1)$ and $\delta_1(Y) = Y$ and $\delta_2(Y) = 0$.

   Note
$$R(\theta, \delta_2) = \mathbb{E}[|\theta - 0|] = |\theta|.$$
   On the other hand (noting $\bar{X}$ has the $N(\theta, 1/25)$ distribution, so that $\theta - \bar{X}$ has the $N(0, 1/25)$ distribution)
$$\begin{aligned} R(\theta, \delta_1) = \mathbb{E}[|\theta - \bar{X}|] &= \int_{-\infty}^{\infty} |x| \frac{5}{\sqrt{2\pi}} e^{-(25/2)x^2} dx \\ &= \frac{5\sqrt{2}}{\sqrt{\pi}} \int_0^{\infty} x e^{-(25/2)x^2} dx \\ &= \frac{\sqrt{2}}{5\sqrt{\pi}} \int_0^{\infty} e^{-u} du = \frac{1}{5}\sqrt{2/5}. \end{aligned}$$