Math 500, Problem Set 1

February 5, 2010

Easier Exercises: Good for the soul, but probably won't bother presenting solutions.

1. A fair coin is thrown repeatedly. What is the probability that on the nth throw:
   - a head appears for the first time
   - the number of heads and tails to date are exactly equal.
   - exactly two heads have appeared altogether to date
   - at least two heads have appeared to date.
2. Let $\mathcal{F}$ and $\mathcal{G}$ be σ-fields of subsets of Ω:
   - Let $\mathcal{H} = \mathcal{F} \cap \mathcal{G}$ be the collection of subsets of Ω lying in both $\mathcal{F}$ and $\mathcal{G}$. Show that $\mathcal{G}$ is a σ field.
   - Show that $\mathcal{F} \cup \mathcal{G}$ need not be a σ field.
   - Show that $\mathcal{F} \times \mathcal{G}$ need not be a σ-field of $\Omega \times \Omega$.
3. Show that if A is independent of itself, show that $\mathbb{P}(A) \in \{0, 1\}$. Also, if $\mathbb{P}(A) = 0, 1$ show that A is independent of all events B.

Medium problems:

1. Let $A_1, A_2, \ldots, A_n$ be events, and let N_k be the event that exactly k of the A_i occur. Prove Waring's Theorem:

$$\mathbb{P}(N_k) = \sum_{i=0}^{n-k} (-1)^i \binom{k+i}{k} S_{k+i} \quad \text{where} \quad S_j = \sum_{i_1 < i_2 < \cdots < i_j} \mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j})$$

2. Let $A_1, A_2, \ldots$ be a sequence of events. Define

$$B_n = \bigcup_{m=n}^{\infty} A_m, \quad C_n = \bigcap_{m=n}^{\infty} A_m$$

Note that $C_n \subseteq A_n \subseteq Bn$, the sequences $\{B_n\}$ and $\{C_n\}$ are decreasing and increasing respectively with limits:

$$\lim B_n = B = \bigcap_n B_n = \bigcap_n \bigcup_{m \geq n} A_m \qquad \lim C_n = C = \bigcup_n C_n = \bigcup_n \bigcap_{m \geq n} A_m.$$

The events B and C are denoted $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$ respectively. Show that

- $B = \{\omega : \omega \in A_n$for infinitely many values of n,
- $C = \{\omega : \omega \in A_n$for all but finitely many values of n.

If $B = C$ we say that $\{A_n\}$ converges to a limit and $\lim A_n = A$. Suppose $A_n \to A$ and show that

- A is an event, in that $A\ in \mathcal{F}$
- $\mathbb{P}(A_n) \to \mathbb{P}(A)$.

3. A biased coin (H/T probabilities p and $1 - p$) is tossed repeatedly. Find the probability that there is a run r of heads in a row, before there is a run of s tails, where r and s are integers.

4. A random subset of $[n](= \{1, \ldots, n\})$ is chosen, selecting each element independently with probability p. Find a (reasonable) upper bound for the probability that there is an arithemetic progression of length k.

5. An urn contains r red balls, and b blue balls where $ab \neq 0$. Balls are removed at random and discarded until the first time that a ball B is removed having a different color from it's predecessor. The ball B is now replaced and the procedure restarted. This process continues until the last ball is drawn from the urn. Show that this lass ball is equally likely to be red or blue.

6. 10 percent of the surface of a sphere is colored red, and the rest blue. Show that, irrespective of the manner in which the colors are distributed, it is possible to inscribe a cube in S with all its vertices red.

7. The event A is said to be repelled by the event B is $\mathbb{P}(A|B) < \mathbb{P}(A)$, and to be attracted by B is $\mathbb{P}(A|B) > \mathbb{P}(A)$. Show that if B attracts A, then A attracts B and B^c repels A. If A attracts B, and B attracts C, does A attract C.

8. Kounias's inequality: Show that

$$\mathbb{P}\left(\bigcup_{r=1}^{n} A_r\right) \leq \min_k \left\{\sum_{r=1}^{n} \mathbb{P}(A_r) - \sum_{r:r\neq k} \mathbb{P}(A_r \cap A_k)\right\}$$

9. Four witnesses A, B, C and D at a trial each speak the truth with probability $\frac{1}{3}$ independently of each other. In their testimonies A claimed that B denied that C declared that D lied. What is the (conditional) probability that D told the truth.

10. n men and n women in n couples are randomly seated around a table so that the men and women alternate. Show that the probability that nobody sits next to their own partner is

$$\frac{1}{n!}\sum_{k=0}^{n}(-1)^k \frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!.$$

Hint: It might be useful to show that the number of ways of selecting k non-overlapping pairs of adjacent seats is $\binom{2n-k}{k}\frac{2n}{2n-k}$..

11. Show that there is no a.a.s. no dominating set in a graph G in $G(n,p)$ of size $m = (1-\epsilon)\frac{\log(np)}{p}$, where $\epsilon > 0$ fixed and $p = \frac{1}{\log^2(n)}$. (Note, setting $p = \frac{1}{\log^2(n)}$ is basically for your conveneince).

12.
    - Show that if G is a graph on n vertices such that the degree of every vertex is at least $\log_2(n)$, then G contains a cycle of even length
    - (*slightly harder) Show that if G is a graph on n vertices such that the degree of every vertex is at least $\log_2(n) - \frac{1}{2}\log_2\log_2(n)$, then G contains a cycle of even length.

13. Prove the following basic (useful) estimates:

    - $$\left(\frac{n}{k}\right)^k < \binom{n}{k} < \left(\frac{en}{k}\right)^k$$
    - $$\binom{n}{\alpha n} < e^{H(\alpha)}$$
      where $H(\alpha) = -\alpha\log(\alpha) - (1-\alpha)\log(1-\alpha)$. (You may want to use Stirlings Formula for this one).
    - $$\binom{2n}{n} = (1+o(1))\frac{4^n}{\sqrt{\pi n}}$$
      Again, you may use Stirlings Formula
    - $$(1-\frac{1}{n})^n < \frac{1}{e} < (1-\frac{1}{n})^{n-1}$$

14. Let $n \geq 2$ and let H be a hypergraph (that is, a collection of sets) all of whose sets have size n. Prove that if H has 4^{n-1} edges, then there is a 4-coloring of the points in H such that none of the sets in H is monochromatic.

15. Let S be a set of binary strings of length $n \geq 2$. Then S is called universal if for each pair $\{i, j\} \subset \{1, 2, \ldots, n\}$ and any binary string ab of length two, there is a string s such that $s_i = a$, and $s_j = b$.

    - Prove that every universal set of strings of length n has size at least $\log_2(n)$.
    - Prove that there is a universal set of binary strings of size at most $8\log(n)$.

16. Let $n \in N$. A maximal chain in $[n]$ is a sequence of subsets $(A_i)_{i=0}^n$ with $A_i \subseteq A_{i+1}$ and $|A_i| = i$. An antichain in $[n]$ is a family of sets where no set is contained in another.

    - How many maximal chains in $[n]$ contain a fixed subset $[n]$ of size r?
    - Let $\mathcal{A}$ be an antichain of subsets of $[n]$. Show, by aking a randomly and uniformly chose maximal chain in $[n]$ that

    $$\sum_{r=0}^{n} \frac{|\mathcal{A}_r|}{\binom{n}{r}} \leq 1 \qquad \text{where } \mathcal{A}_r = \{A \in \mathcal{A} : |A| = r\}.$$

    - Use the last part to show that $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$.
    - Show that there exists a set $A \subseteq [n]$ such that A is missing no 3 consecutive elements (that is one of $\{k, k+1, k+2\} \in A$, but that the longest arithmetic progression in A has size $O(\log(n))$.