Math 500, Problem Set 1

March 3, 2010

1. Prove that for every $\epsilon > 0$ there is a finite $l_0 = l_0(\epsilon)$ and an infinite sequece of bits $a_1, a_2, \ldots$ with $a_i \in \{0,1\}$, such that for every $l > l_0$ and every $i \geq 1$, the two binary vectors $u = (a_i, a_{i+1}, \ldots, a_{i+l-1})$ and $v = (a_{i+l}, a_{i+l+1}, \ldots, a_{i+2l-1})$ differ in at least $(\frac{1}{2} - \epsilon)l$ coordinates.

2. Let $G = (V, E)$ be a simple graph, and suppose each $v \in V$ is associated with a set $S(v)$ of colors of size at least $10d$, where $d \geq 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$ there are at most d neighbors u of v such that c lies in $S(u)$. Prove there is a proper coloring of G assigning to each vertex v a color from its class $S(v)$.

3. Let X be a random variable which takes non-negative values only. Show that
$$\sum_{i=1}^{\infty}(i-1)1_{A_i} \leq X \leq \sum_{i=1}^{\infty} i1_{A_i},$$
where $A_i = \{i - 1 \leq X < i\}$. Deduce that
$$\sum_{i=1}^{\infty}\mathbb{P}(X \geq i) \leq \mathbb{E}[X] \leq 1 + \sum_{i=1}^{\infty}\mathbb{P}(X \geq i).$$

4. The interval [0,1] is partitioned into n disjoint sub-intervals with lengths $p_1, p_2, \ldots, p_n$, and the *entropy* of this partition is defined to be
$$h = -\sum_{i=1}^{n} p_i \log p_i.$$
Let $X_1, X_2, \ldots$ be independent random variables having uniform distribution on $[0,1]$, and let $Z_m(i)$ be the number of $X_1, \ldots, X_m$ which like in the ith interval of the partition above. Show that
$$R_m = \prod_{i=1}^{n} p_i^{Z_m(i)}.$$
satisfies $m^{-1}\log(R_m) \to -h$ a.s. as $m \to \infty$ (that is:
$$\mathbb{P}(\lim_{m\to\infty}\frac{\log(R_m)}{m} = -h) = 1.$$

5. Let $A_1, A_2, \dots$ be a sequence of events. Show that

$$\mathbb{P}(A_n \ \ i.o.) \geq \limsup_{n\to\infty} \mathbb{P}(A_n).$$

6. Suppose $X_1, X_2, \dots$ are independent and exponentially distributed with paratmeter one. Show that

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{X_n}{\log(n)} = 1\right) = 1$$

7. Let X be a random variable with $\mathbb{E}[X] = 0$ and $\mathrm{Var}(X) = \sigma^2$. Prove that for all $\lambda > 0$,

$$\mathbb{P}(X \geq \lambda) \leq \frac{\sigma^2}{\sigma^2 + \lambda^2}.$$

   (Note: this is a slight improvement on Chebyshev, which gives a bound of the form $\frac{\sigma^2}{\lambda^2}$.)

8. Let X be a random variable taking integral nonnegative values, let $\mathbb{E}[X^2]$ denote the expectation of its square, and let $\mathrm{Var}(X)$ denote its variance. Prove

$$\mathbb{P}(X = 0) \leq \frac{\mathrm{Var}(X)}{\mathbb{E}[X^2]}.$$

9. Show that there is a positive constant c such that the following holds. For any n vectors $a_1, a_2, \dots, a_n \in \mathbb{R}^2$ satisfying $\sum_{i=1}^n ||a_i||^2 = 1$ and $||a_i|| \leq \frac{1}{10}$, where $||\cdot||$ denotes the usual Euclidean norm, if $(\epsilon_1, \dots, \epsilon_n)$ is a $\{-1, 1\}$-random vector obtained by choosing each ϵ_i randomly and independently with uniform distribution to be either -1 or 1, then

$$\mathbb{P}\left(\left\|\sum_{i=1}^n \epsilon_i a_i\right\| \leq \frac{1}{3}\right) \geq c.$$

10. Let $S_n = \sum_{i=1}^n X_i$ where the X_i are iid variables that are uniformly ± 1.

    (a) Show that

$$\mathbb{P}\left(\sqrt{\left(\frac{S_n}{\sqrt{n}}\right)^2} \geq \frac{1}{3}\right) \geq c$$

    for some absolute constant c. (This is a special case of the previous problem.)

    (b) Show that

$$\mathbb{E}[|S_n|] = n2^{1-n}\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}.$$

    Apparently a 1974 Putnam problem.

(c) Derive that

$$\mathbb{E}[|S_n|] \sim (\sqrt{\frac{2}{\pi}} + o(1))\sqrt{n}.$$

11. Let $X_2, X_3, \ldots$ be independent random variables such that:

$$\mathbb{P}(X_n = n) = \mathbb{P}(X_n = -n) = \frac{1}{2n \log n}, \quad \mathbb{P}(X_n = 0) = 1 - \frac{1}{n \log n}.$$

Show that this sequence obeys the weak law but not the strong law, in the sense that $n^{-1} \sum X_i$ converges to zero in probability but not almost surely.

12. Construct a sequence $\{X_r : r \geq 1\}$ of independent random variables with zero mean such that $n^{-1} \sum_{r=1}^{n} X_r \to -\infty$ almost surely, as $n \to \infty$.

13. Prove that $\mathbb{P}(X = 0) \leq \frac{\sigma^2}{\mu^2}$ when X has mean $\mu \neq 0$ and variance σ^2 (essentially, we did this in class.) Then show that the stronger inequality below is true:

$$\mathbb{P}(X = 0) \leq \frac{\sigma^2}{\mu^2 + \sigma^2}.$$