1. (*25 points*) Suppose $X_1, X_2, \ldots, X_n$ is a sample from a $Expo(1/\theta)$ distribution, and let $Y_1, \ldots, Y_n$ denote the order statistics of the sample. **a.** (*10 pts*) Find the constant c so that cY_1 is an unbiased estimator of θ.

$f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$ $\quad F(x,\theta) = \int_0^x \frac{1}{\theta} e^{-x/\theta} = 1 - e^{-x/\theta}$

$f_{Y_1}(y) = \frac{n}{\theta} e^{-(n/\theta)x}$ $\quad \mathbb{E}[Y_1] = \frac{\theta}{n}$ $\quad E[aY_1] = \theta$

$Y_1 \sim expo\left(\frac{n}{\theta}\right)$

$\boxed{c = n}$

b. (*10 pts*) Using the knowledge that $Z = \sum X_i$ is an sufficient statistic for θ, find an MVUE for θ. Explain why you know that you estimator is an MVUE. (You need not use Part (a)).

$\mathbb{E}[X_i] = \theta,$

$\mathbb{E}\left[\frac{Z}{n}\right] = \theta$, as this is a function of Z; sufficient (and complete as X_i is a reg. exponential family) and unbiased this is an MVUE.

c. (*5 pts*) Compute the variance of the estimators you found in (*a*) and (*b*).

$Var(nY_1) = n^2 Var(Y_1) = n^2 \frac{\theta^2}{n^2} = \theta^2.$

$Var\left(\frac{Z}{n}\right) = \frac{1}{n^2} Var(\sum X_i) = \frac{\theta^2}{n}.$

2. (*25 points*) **a.** (*15 pts*) Suppose $X_1, \ldots, X_n$ are $N(\mu, \sigma)$ where both μ and σ are unknown. Supposing $\bar{X} = 1$, $S^2 = 2.5$ and $n = 100$, construct an exact 95% confidence interval for σ^2.

Recall: $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$ in this case.

$$\therefore P\left(\chi^2_{.025}(99) \le \frac{(n-1)S^2}{\sigma^2} \le \chi^2_{.975}(99)\right) = .95$$

So:

$$\left(\frac{99 \cdot 2.5}{\chi^2_{.975}(99)} \le \sigma^2 \le \frac{99(2.5)}{\chi^2_{.025}(99)}\right)$$

is a 95% confidence interval for σ^2

b. (*10 pts*) Suppose $X_1, \ldots, X_n$ are a sample from an unknown distribution. If $\bar{X} = 1$, $S^2 = 2.5$ and $n = 100$, construct an approximate 95% confidence interval for $\sigma^2 = \mathrm{Var}(X_i)$.

Actually, follows same way only now not exact

$$\left(\frac{99 \cdot 2.5}{\chi^2_{.975}(99)}, \frac{99(2.5)}{\chi^2_{.25}(99)}\right)$$

I meant to make this more interesting - but didn't.

3. (*15 points*) A Weibell distribution is a distribution with $f(x) = \frac{1}{\theta^3} 3x^2 e^{-x^3/\theta^3}$ for $0 < x < \infty$. Suppose we know how to generate uniform $(0,1)$ random variables U, explain how to generate a random variable with the Weibell distribution using U.

$$F(x) = \int_0^x \frac{1}{\theta^3} 3t^2 e^{-t^3/\theta^3}\, dt = \int_0^{x^3/\theta^3} e^{-u}\, du = -e^{-u}\Big|_0^{x^3/\theta^3}$$

$$u = t^3/\theta^3 \qquad = 1 - e^{-x^3/\theta^3}$$

$$du = \frac{3t^2}{\theta^3}$$

$$y = 1 - e^{-x^3/\theta^3}$$

$$e^{-x^3/\theta^3} = 1 - y$$

$$-x^3/\theta^3 = \ln(1-y)$$

$$\frac{x}{\theta} = \ln\left(\frac{1}{1-y}\right)^{\frac{1}{3}}$$

$$x = \theta \ln\left(\frac{1}{1-y}\right)^{\frac{1}{3}}$$

$\therefore\ X = \theta \ln\left(\frac{1}{1-U}\right)^{\frac{1}{3}}$ has Weibull distribution

4. (*30 points*) Let $X_1, \ldots, X_n$ have the Poisson(θ) distribution. **a.** (*20 pts*) Show $\bar{X}$ is an efficient estimator of θ. (You may use the fact that $\text{Var}(\bar{X}) = \theta/n$, and need not compute it.)

$$f(x;\theta) = \frac{e^{-\theta}\theta^{k}}{k!}$$

$$\log(f(x;\theta)) = -\theta + k\log(\theta) - \log(k!)$$

$$\partial\theta = -1 + \frac{k}{\theta}$$

$$\partial^2\theta = -\frac{k}{\theta^2}$$

$$I(\theta) = -E[\partial^2\theta] = \frac{1}{\theta}$$

$$k(\theta) = E[\bar{X}] = \theta \qquad k'(\theta) = 1$$

R-C Lower bound

$$\frac{k'(\theta)}{nI(\theta)} = \frac{1}{n \cdot \frac{1}{\theta}} = \frac{\theta}{n} = \text{Var}(\bar{X}),$$ so this is efficient

b. (*10 pts*) Show that $\bar{X}$ is a complete and sufficient statistic for θ.

$$f(x;\theta) = e^{\underset{p(\theta)}{-\theta} + \underset{K(x)}{x}\,\underset{q(\theta)}{\log(\theta)} - \underset{f(x)}{\log(k!)}}$$

This is a regular exponential family, so $\sum X_i = \sum K(x)$ is complete / sufficient.

$\therefore$ so is $\frac{\sum X_i}{n} = \bar{X}$.

5. (*25 points*) Let $X_1, \ldots, X_n$ be a random sample from a $\Gamma(\alpha, beta)$ distribution where α is known and β is not. **a.** (*10 pts*) Determine the likelihood ratio test for $H_0 : \beta = \beta_0$, against $H_1 : \beta \neq \beta_0$. In particular, suppose $Z = -2\log\Lambda$. For what values of Z should we reject H_1 and accept H_0 if we want a test with approximate size .99.

$$-2\log\Lambda \approx \chi^2(1)$$

$\therefore$ Reject if $Z \geq 6.635$

b. (*10 pts*) Find a uniformly most powerful test for $H_0 : \beta = \beta_0$ versus $H_1 : \beta > \beta_0$.

$$f(x;\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} \exp(-x/\beta)$$

$$\frac{L(x;\beta_0)}{L(x;\beta_1)} = \left(\frac{\beta_0}{\beta_1}\right)^{\alpha n} \exp\left(-\frac{\sum x}{\beta_0} + \frac{\sum x}{\beta_1}\right)$$

This is a decreasing function of $\sum X$ for any $\beta_1 > \beta$ so uniformly most powerful test is accept H_1 when $\sum X_i \geq c$.

c. (*5 pts*) Is there a uniformly most powerful test of $H_0 : \beta = \beta_0$ versus $H_1 : \beta \neq \beta_0$? Why or why not?

No: $-\frac{\sum X_i}{\beta_0} + \frac{\sum X}{\beta_1}$ is negative if $\beta_1 > \beta_0$, positive if $\beta_1 < \beta_0$

$\therefore$ Best test is: $\sum X_i \geq c$ if $\beta_1 > \beta_0$

$\sum X_i \leq c$ if $\beta_1 < \beta_0$

$\therefore$ No uniformly most powerful test

6. (*25 points*) Suppose $X_1, \ldots, X_n$ are a sample from the following distributions. Find an mle $\hat{\theta}$ of θ. **a.** (*10 pts*) $f(x;\theta) = (1/\theta)e^{-x/\theta}$, $0 < x < \infty$, and $0 < \theta < \infty$, zero elsewhere.

$$L(\theta) = \left(\frac{1}{\theta}\right)^n e^{-\sum x_i/\theta}$$

$$\ell(\theta) = -n \log(\theta) - \frac{\sum X_i}{\theta}$$

$$\ell'(\theta) = \frac{-n}{\theta} + \frac{\sum X_i}{\theta^2}$$

$$\ell'(\theta) = 0 \Rightarrow n\theta = \sum X_i$$

$$\hat{\theta} = \bar{X}.$$

b. (*15 pts*) $f(x;\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{1}{2\theta}x^2}$, for $-\infty < x < \infty$ and where $1 \le \theta \le 2$.

$$L(\theta) = \left(\frac{1}{2\pi\theta}\right)^{n/2} e^{-\frac{1}{2\theta}\sum X_i^2}$$

$$\ell(\theta) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\theta) - \frac{1}{2\theta}\sum X_i^2$$

$$\ell'(\theta) = -\frac{n}{2\theta} + \frac{1}{2\theta^2}\sum X_i^2$$

$$\ell'(\theta) = 0 \Rightarrow \frac{n}{2}\theta = \frac{1}{2}\sum X_i^2$$

$$\Rightarrow \hat{\theta} = \frac{\sum X_i^2}{n}$$

7. (*25 points*) Let $X_1, \ldots, X_n$ denote a random sample from a distribution of pdf $\theta e^{-\theta x}$, for $0 < x < \infty$ and zero elsewhere, and $\theta < 0$. $\sum x_i$ is a sufficient statistic (and complete) for θ. Show $(n-1)/Y$ is the MVUE of θ.
Hint: What is the distribution of $\sum X_i$.

$$X_i \sim \exp(\theta) = \Gamma\left(1, \tfrac{1}{\theta}\right)$$

$$Y \sim \Gamma\left(n, \tfrac{1}{\theta}\right)$$

$$E\left[\frac{1}{Y}\right] = \cancel{\theta} \int_0^\infty \frac{\theta^{n-1}}{(n-1)!} y^{n-2} e^{-y/\theta} \, dy$$

$$= \frac{\theta (n-2)!}{(n-1)!} \cdot \underbrace{\int_0^\infty \frac{\theta^{n-2}}{(n-2)!} y^{n-2} e^{-y/\theta} \, dy}_{1 \text{ as integrating pdf of } \Gamma\left(n-1, \frac{1}{\theta}\right)}$$

$$= \frac{\theta}{n-1}.$$

$\therefore$ $E\left[\frac{n-1}{Y}\right] = \theta$, so this is an unbiased estimator which is a function of a complete and sufficient statistic, $\therefore$ is an MVUE.

8. (*25 points*) The *Pareto distribution* has CDF

$$F(x;\theta_1,\theta_2) = \begin{cases} 1-(\theta_1/x)^{\theta_2} & x \geq \theta_1 \\ 0 & else. \end{cases}$$

Find the mles of θ_1 and θ_2.
Note: I gave you the CDF!

$$f(x;\theta_1;\theta_2) = \theta_2 \left(\frac{\theta_1}{x}\right)^{\theta_2 - 1} \cdot \frac{\theta_1}{x^2}$$

$$= \theta_2 \frac{\theta_1^{\theta_2}}{x^{\theta_2+1}}$$

$$L(x;\theta_1,\theta_2) = \begin{cases} \dfrac{\theta_2^n \theta_1^{\theta_2 n}}{(\prod x_i)^{\theta_2+1}} \\ 0 & \text{if } \min(x_i) > \theta_1 \end{cases}$$

$\therefore L(x;\theta_1;\theta_2)$ increasing function of θ_1, $\therefore$ $\hat{\theta}_1 = \min(X_i)$

$$l(x;\theta_1;\theta_2) = n\log(\theta_2) + n\theta_2 \log(\theta_1) - (\theta_2+1)\log(\prod X_i)$$

$$\frac{\partial}{\partial\theta_2} l'(x;\theta_1;\theta_2) = \frac{n}{\theta_2} + n\log(\theta_1) - \log(\prod X_i)$$

$$\therefore \hat{\theta}_2 = \frac{-n}{n\log(\theta_1) - \log(\prod X_i)} = \frac{n}{\sum(\log(x_i) - \log(\theta_1))}$$