

DEGREE SUM AND VERTEX DOMINATING PATHS

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ABSTRACT. A *vertex dominating path* in a graph is a path P such that every vertex outside P has a neighbor on P . In 1988 H. Broersma [HB88] stated a result implying that every n -vertex k -connected graph G such that $\sigma_{(k+2)}(G) \geq n - 2k - 1$ contains a dominating path. We provide a short, self-contained proof of this result and further show that every n -vertex k -connected graph such that $\sigma_2(G) \geq \frac{2n}{k+2} + f(k)$ contains a dominating path of length at most $O(|T|)$, where T is a minimum dominating set of vertices. The main result is a proof that every n -vertex k -connected graph such that $\sigma_2(G) \geq \frac{2n}{k+2} + f(k)$ contains a path of length at most $O(|T|)$, through any set of T vertices where $|T| = o(n)$.

1. INTRODUCTION

Interest in dominating cycles and paths of various sorts began as a natural relaxation of hamiltonian cycle and path problems and moved in a number of directions: edge dominating cycles (paths), vertex dominating cycles (paths), longest cycles (paths) that dominate in some manner and so forth.

In particular, a paper by Bondy and Fan ([BF87]) proving a conjecture of Clark, Coburn and Erdős [CCE], gave a condition for degree sums of sets of $k + 1$ independent vertices in k -connected graphs that imply the existence of a vertex dominating cycle. Shortly thereafter Broersma ([HB88]) produced a general result providing generalized degree sum conditions in k -connected graphs forcing all vertices to be within a fixed distance of a cycle. Furthermore, at the end of the paper, an analogue for paths is stated. There has continued to be investigation into vertex dominating cycle structures though it has tended to focus on long cycles. (See [MOS], [SY].)

On the other hand, a vertex dominating path may also be viewed as a spanning tree of a particular type, sometimes called a caterpillar. There has been much recent work on conditions implying particular structural properties in spanning trees. See for example [FKKLR], [CFHJL], or the recent survey by Ozeki and Yamashita ([OY11]) on spanning trees.

A converging of these two streams of research occurred in a recent paper by Faudree, Gould, Jacobson, and West ([FGJW]) which contains several theorems relating minimum degree and vertex dominating paths (or spanning caterpillars). Motivated by [FGJW], this paper contains results relating degree sum conditions and dominating paths. Included is a short, self-contained proof of a result originally stated in [HB88] and a theorem with corollary that answers a question from [FGJW].

All graphs are finite and simple. Notation and terminology generally follows West [DW96]. A set $X \subset V(G)$ *dominates* the graph G if every vertex of $G - X$ has a neighbor in X . Observe that this definition means the set X dominates the vertex set but not necessarily the edge set of G . We will often say a path P or a cycle C dominates G if $V(P)$ or $V(C)$ dominates $V(G)$.

Given a graph G and integer $k \geq 2$, we denote by $\sigma_k(G)$ the minimum degree-sum independent sets of k vertices. Observe that, by a natural extension of this definition, $\sigma_1(G) = \delta(G)$.

Given a set of vertices $X = \{x_1, x_2, \dots, x_m\} \subseteq V(G)$, the notation $N[X] = X \cup \{v \in V(G) \mid \exists x \in X, vx \in E(G)\}$, often called the *closed neighborhood of X* .

2. DEGREE SUM AND THE EXISTENCE OF A DOMINATING PATH

In 1985, Clark, Colbourn, and Erdős [CCE] conjectured that every k -connected graph with minimum degree at least $\frac{n}{k+1} + f(c)$ has a vertex dominating cycle. In [BF87], Bondy and Fan proved the conjecture holds and made another similar conjecture replacing degree sum with distance m neighborhoods and replacing dominating cycles with m -dominating cycles (where a cycle is m -dominating if every vertex is at most distance m from the cycle.) In [HB88], Broersma generalized the classic Erdős-Chvátal condition for hamiltonicity and, as one part of one corollary, settled the conjecture of Bondy and Fan. At the end of the paper by Broersma, several general results concerning paths, all of which are analogues of the cycle results proved earlier, are stated. These earlier proofs are intricate, nontrivial, and carefully linked. Below is one corollary. Note that a Δ_λ -traceable graph is one in which there exists a path such that all vertices are a distance at most λ from the path and a set of *mutually r -distant vertices* is a set $S \subseteq V(G)$ such that for every $u, v \in S, u \neq v$, the distance from u to v is at least r .

Corollary 2.1. [HB88] *Let G be an n -vertex, k -connected graph ($k \geq 1$) and let $\lambda \geq 2$. If the degree sum of any $k + 2$ mutually $(2\lambda - 1)$ -distant vertices is at least $n - 2k - 1 - (\lambda - 2)k(k + 2)$, then G is Δ_λ -traceable.*

We offer a self-contained proof of the special case of Corollary 2.1 when $\lambda = 2$ and the path is vertex-dominating.

Theorem 2.1. *Every n -vertex k -connected graph with $\sigma_{(k+2)}(G) \geq n - 2k - 1$ contains a vertex-dominating path.*

Proof. Let G satisfy the hypothesis of the theorem and proceed by contradiction. Let P be a path in G dominating a maximum number of vertices. Further, among all paths dominating a maximum number of vertices, choose P to be as short as possible. Label the vertices of P as: $x = x_1, x_2, \dots, x_t = y$. If all the neighbors of x (or y) were dominated by the vertices of $P - x$ (or $P - y$), then a shorter path dominating the same number of vertices is possible. Thus there exist vertices x' (and y') such that $N(x') \cap V(P) = x$ (and $N(y') \cap V(P) = y$). In this case, we will say x' is *uniquely dominated* by x or that x *uniquely dominates* x' . Clearly this relationship depends upon the choice of P , but we are assuming the choice of P is fixed.

Since G has no vertex dominating path, there exists a vertex, z , not dominated by P . Since G is k -connected, there exist k paths from z to P that are vertex disjoint other than at initial vertex z . Pick these k paths to as short as possible and label the terminal vertices $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ in order as they appear on P from x to y .

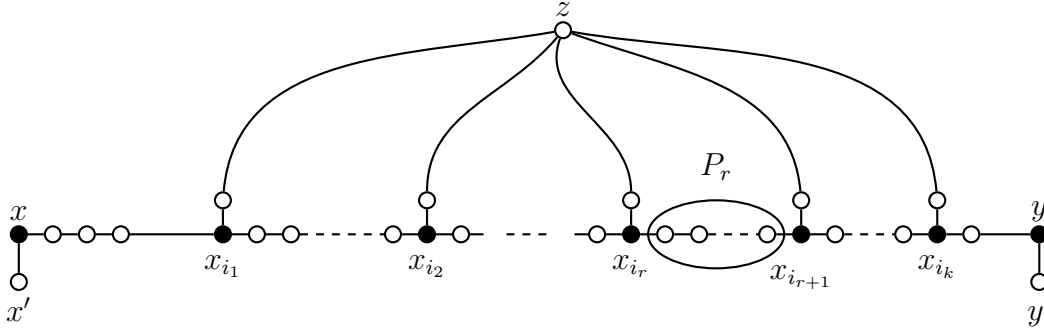


FIGURE 1. This figure illustrates the paths from vertex z to P , a shortest path dominating a maximum number of vertices. Note the circled section denotes P_r , the section of the path strictly between two consecutive paths, x_r and x_{r+1} .

Let $P_r = P(x_{i_r}, x_{i_{r+1}})$ denote the subpath of P strictly between two consecutive endpoints of paths from z . (See Figure 1.)

A vertex, u , in a subpath P_r is called *moveable* if either of the criteria below are satisfied:

- (1) u has consecutive adjacencies on $P - P_r$. (That is, u could be “inserted” elsewhere on $P - P_r$.)
- (2) Both u and all of its neighbors have adjacencies on $P - P_r$. (That is, u and all vertices it dominates are dominated elsewhere on $P - P_r$.)

If there exists a single subpath such that all vertices on it are moveable, then a path dominating more vertices can be obtained by replacing this subpath with a path through z . Thus, for each of the $k - 1$ subpaths, there exists at least one vertex that is not moveable. Label as u_r the first vertex in P_r that is not moveable as P_r is traversed from x_{i_r} to $x_{i_{r+1}}$.

Let Q_r consist of all the vertices on P_r between x_{i_r} and u_r . Observe that, for $r_1 \neq r_2$, there are no edges between Q_{r_1} and Q_{r_2} and no common neighbors in $V - V(P)$. If any such edges or paths existed, choosing the “first” one (that is, the one with smallest index on P) would produce a path that dominates all the vertices that P dominates and z . The importance of this observation is the conclusion that all vertices of Q_{r_1} are not only moveable, but are moveable to regions of P other than Q_{r_2} as is the case for the vertices dominated by Q_{r_1} . (See Figure 2.)

The fact that u_r is not moveable implies either u_r dominates a vertex, u_r^* , that is not dominated by any vertex outside P_r (i.e. $N_P(u_r^*) \subseteq V(P_r)$), or u_r itself is not dominated outside of P_r (i.e. $N_P(u_r) \subseteq V(P_r)$.)

Now define a set of vertices S to contain x' , y' , and z . Furthermore, for every subpath, add either u_r^* or u_r to S , using whichever one has no neighbors on $P - P_r$.

By definition, $|S| = k + 2$, S is independent, and no pair of vertices in S can have a common neighbor on P . Moreover, if any pair of vertices in S has any common neighbor outside of P a path dominating more vertices can be found. Specifically, this path can be built including z , along with the two involved vertices of S and relies on the fact that vertices of Q_{r_1} and Q_{r_2} can be moved if

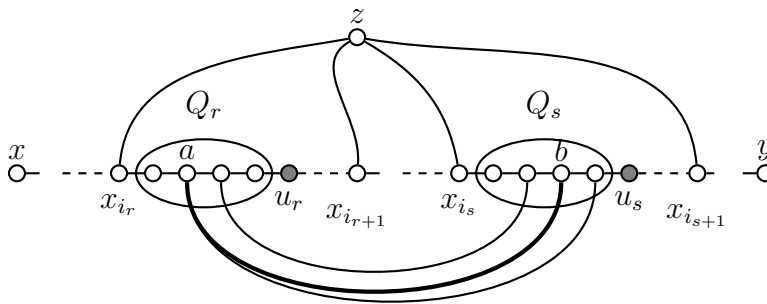


FIGURE 2. For each subpath of P , u_i is the first *nonmovable* vertex and, so, those in Q_i are all movable. Observe that any edges between distinct Q_i 's results in a path dominating more vertices by using the edge between vertices of smallest indices (or, alternatively, left-most vertices). New path: x to x_{i_r} to z to x_{i_s} to a to b to y . Note that all vertices of Q_r and Q_s not on the new path are dominated elsewhere.

needed. Thus, for every $a, b \in S$ such that $a \neq b$, $N(a) \cap N(b) = \emptyset$. Since none of the vertices in S are adjacent to $\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}$, we have $\sigma_{k+2}(G) \leq \sum_{a \in S} d(a) \leq n - 2k - 2$, a contradiction. \square

The following example shows that the preceding theorem is best possible.

Example 2.1. Construct an n -vertex k -connected graph G as follows. Begin with a complete graph on k vertices, K_k , and $k + 2$ independent vertices, v_1, v_2, \dots, v_{k+2} . Partition the remaining $n - 2k - 2$ vertices into $k + 2$ complete graphs such that the orders of the graphs are as equal as possible and label them B_1, B_2, \dots, B_{k+2} . For each i add all edges between B_i and $K_k \cup v_i$. This graph has no dominating path and $\sigma_k(G) = n - 2k - 2$. Note that $n \geq k^2 + 4k + 2$.

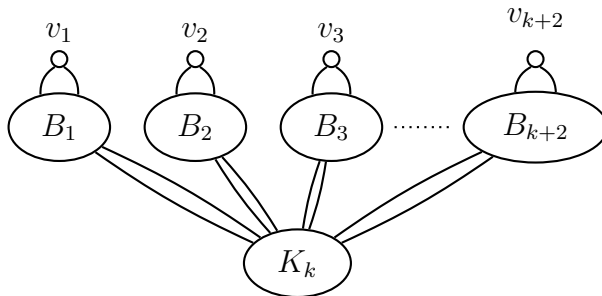


FIGURE 3. Example 2.1

The following corollary follows immediately from the statement of the theorem:

Corollary 2.2. Let $r \geq 1$ be an integer. Every n -vertex k -connected graph with $\sigma_r(G) \geq r \left(\frac{n-2k-1}{k+2} \right)$ contains a vertex-dominating path.

The following corollary follows immediately from the proof of the theorem:

Corollary 2.3. If $\alpha(G) \leq \kappa(G) + 2$, then G contains a vertex dominating path.

3. PRELIMINARY LEMMAS

In order to streamline the proof of the results in Section 4, several preliminary results are presented. Below is a well-known result by Dirac:

Lemma 3.1. [GD52] *If G is k -connected and $X \subseteq V(G)$ such that $|X| \leq k$, then there exists a cycle in G containing all the vertices of X .*

Lemma 3.2, which asserts the existence of small dominating sets relative to a minimum degree condition, is used to prove Lemma 3.3, an analogous result for degree sum conditions.

Lemma 3.2. *Every n vertex graph G with minimum degree $\delta \geq \beta n$ where $0 < \beta < 1$ contains a dominating set $X \subseteq V(G)$ such that $|X| \leq \lceil \log_{1/(1-\beta)} n \rceil$.*

Proof. Let G be an n vertex graph with minimum degree $\delta \geq \beta n$ where $0 < \beta < 1$. The proof will proceed by iteratively constructing a dominating set of vertices using no more than $\lceil \log_{1/(1-\beta)} n \rceil$ vertices.

Begin by choosing a vertex x_1 . Let $X = \{x_1\}$ and define $S_1 = V(G) - N[x_1]$, the set of vertices not dominated by x_1 . Given the iteratively constructed set $X = \{x_1, x_2, \dots, x_i\}$, define $S_i = V(G) - N[X]$, the set of vertices not dominated by X .

We claim there exists a vertex x_{i+1} such that $d_{S_i}(x_{i+1}) \geq \beta|S_i|$ and this is the vertex we will add to X .

If $d_{S_i}(x) < \beta|S_i|$ for every vertex x in G , then, in \overline{G} , $d_{S_i}(x) > (1 - \beta)|S_i|$. Using the previous inequality and the definition of S_i , we conclude $\sum_{v \in V(\overline{G})} d_{\overline{G}}(v) \geq (1 - \beta)|S_i|(n - i) + i|S_i|$. An application of the Pigeon Hole Principle implies there exists a vertex $y \in S_i$ such that $d_{\overline{G}}(y) > (1 - \beta)(n - i) + i$. Thus, $d_G(y) < n - [(1 - \beta)(n - i) + i] = \beta n - i\beta$, contradicting the minimum degree condition.

We claim that in $r = \lceil \log_{1/(1-\beta)} n \rceil$ iterations, the set $|S_r| < 1$ and so X dominates $V(G)$.

By construction $|S_{i+1}| < (1 - \beta)|S_i| < (1 - \beta)^{i+1}n$. For $r > \log_{1/(1-\beta)} n$, $|S_r| < 1$. Thus, the dominating set X requires at most $r = \lceil \log_{1/(1-\beta)} n \rceil$ vertices. \square

The next Lemma, an analogue of Lemma 3.2, uses the same proof technique.

Lemma 3.3. *Every n vertex graph G with $\sigma_2(G) \geq 2\beta n$ where $0 < \beta < 1$ contains a dominating set $X \subseteq V(G)$ such that $|X| \leq \lceil \log_{1/(1-\beta)} n \rceil$.*

Proof. Let G be an n vertex graph with $\sigma_2 \geq 2\beta n$ where $0 < \beta < 1$. The proof will proceed by iteratively constructing a dominating set of vertices using no more than $\lceil \log_{1/(1-\beta)} n \rceil$ vertices.

If $\delta(G) \geq \beta n$, then apply Lemma 3.2. Otherwise, choose a pair of nonadjacent vertices vertex x_1 and x_2 such that $d(x_1) < \beta n$. Let $X = \{x_1, x_2\}$ and define $S_2 = V(G) - N[x_1, x_2]$, the set of vertices not dominated by x_1 or x_2 . Observe that for every $x \in S_2$, $d(x) > \beta n$ because x and x_1

are nonadjacent. Also, note that $|S_2| < (1 - 2\beta)n < (1 - \beta)^2n$. Given the iteratively constructed set $X = \{x_1, x_2, \dots, x_i\}$, define S_i to be the set of vertices not dominated by X .

The proof now follows precisely the same argument and the same arithmetic as Lemma 3.2. \square

The next lemma provides conditions under which the existence of some cycle through a specified set of vertices implies the existence of a small cycle through the specified set.

Lemma 3.4. *Let G be an n -vertex graph with $\sigma_2(G) > \frac{2n}{k+2}$. Let $X \subseteq V(G)$ such that $|X| \leq g(n) = o(n)$. If there exists a cycle in G containing all vertices of X , then, for n sufficiently large, a smallest cycle containing all vertices of X has at most $(3k + 5)|X|$ vertices.*

Proof. Let G and X satisfy the hypothesis of the Lemma and let C be a smallest cycle of G containing X . Proceed by contradiction and assume $|V(C)| > (3k + 5)|X|$. Then there exists a segment of C between consecutive vertices of X with at least $3k + 5$ vertices strictly between them, say $x_1 = v_0, v_1, v_2, \dots, v_q = x_2$, where $q \geq 3k + 6$, $x_1, x_2 \in X$ and $v_1, v_2, \dots, v_{q-1} \notin X$. Let $S = \{v_0, v_3, v_6, \dots, v_{3k+6}\}$.

Because C was chosen to be as small as possible, the set S is independent and no two vertices of S can have a common neighbor in $G - C$. Since $\frac{1}{2} \sum_{i=0}^{k+2} (d(v_{3i}) + d(v_{3(i+1)})) > \frac{1}{2}(k+3) \left(\frac{2n}{k+2}\right) > n$ (where indices are taken modulo $3k + 6$), the vertices of S must have adjacencies on C .

Let $b = n - |V(C)|$. Then the number of chords in C from vertices of S is at least $\frac{(k+3)n}{k+2} - b - 2(k+3)$. Observe that all of these chords have their other endpoint outside the segment of C containing S . Let c count the number of vertices at the other end of the chords from S . Now, $b + c + 3k + 7 \leq n$.

After allowing one chord to each of the c vertices, there remain at least $\frac{(k+3)n}{k+2} - b - 2k - 6 - c$ "excess" chords. Assume a particular segment of C has t vertices with at least one excess chord. Observe that no pair of chords from S to this segment can cross due to the choice of C as smallest possible. (See Figure 4.) Thus, consecutive pairs of the t vertices, taken in the order they appear on C , can share at most one neighbor in S . Thus, these t vertices can accept at most $2k + 3 + (t - 1)$ chords and, therefore, at most $2k + 2$ excess chords. Thus, the total number of excess chords is at most $(2k + 2)|X| = o(n)$.

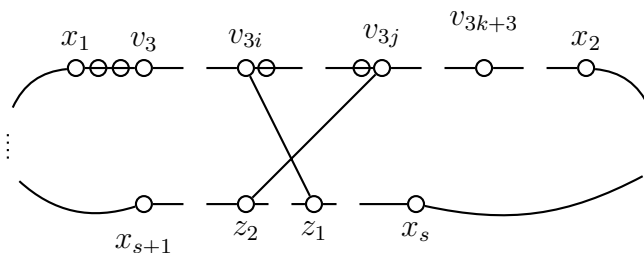


FIGURE 4. A single pair of crossing edges results in a smaller cycle. Follow $x_1 = v_0$ to v_{3i} , down to z_1 , around to v_{3j} via x_s , down to z_2 and back to x_1 .

Now, using the previous upper and lower bounds on the number of excess chords, we produce the following contradiction:

$$\begin{aligned}
 (1) \quad \frac{1}{k+2}n &= \frac{(k+3)n}{k+2} - n \\
 &\leq \frac{(k+3)n}{k+2} - (b+c+3k+7) \\
 &< \frac{(k+3)n}{k+2} - b - 2k - 6 - c \\
 &\leq (2k+2)|X| \\
 &= o(n).
 \end{aligned}$$

Thus, C can have at most $(3k+5)|X|$ vertices. \square

This next lemma gives conditions under which there exists a short path between two specified vertices avoiding a specified set of vertices.

Lemma 3.5. *Let G be an n -vertex graph such that $\sigma_2(G) \geq \frac{2n}{k+2}$. Let $Y \subseteq V(G)$ such that $|Y| \leq g(n) = o(n)$ and $u, v \in V(G) - Y$. If there exists a u, v -path in $G - Y$, then, for n sufficiently large, there exists a Y -avoiding u, v -path using at most $3k+6$ vertices.*

Proof. Let $G, Y, u,$ and v satisfy the hypotheses of the Lemma. Proceed by contradiction and assume that the shortest Y -avoiding u, v -path contains at least $3k+7$ vertices, labeled $u = w_0, w_1, \dots, w_q = v$, where $q \geq 3k+6$. Thus the set $\{w_0, w_3, \dots, w_{3k+6}\}$ is not only an independent set of vertices, but has the property that any pair of vertices from this set have disjoint neighborhoods in $G - Y$. Thus,

$$(2) \quad (k+3) \left(\frac{2n}{k+2} \right) \leq \sum_{i=0}^{k+2} (d(w_{3i}) + d(w_{3(i+1)})) \leq 2(k+3)|Y| + 2(n - |Y|),$$

where $w_{3k+9} = w_0$. For n sufficiently large, equation 2 produces a contradiction. Thus, a smallest Y -avoiding u, v -path has at most $3k+6$ vertices. \square

Lemma 3.6. *Let G be an n -vertex graph such that $\sigma_2(G) \geq \frac{2n}{k+2}$ where $k \geq 1$ is an integer. Let $T \subseteq V(G)$ such that $|T| \leq g(n) = o(n)$. If $\kappa(G) \geq 9k|T|$, then, for n sufficiently large, there exists a cycle (or path) in G on at most $9k|T|$ vertices containing all the vertices of T in any order.*

Proof. Arbitrarily order the vertices of T : x_1, x_2, \dots, x_t . Since G is connected, Lemma 3.5 implies there exists an x_1, x_2 -path on at most $9k$ vertices. Call it $P_{1,2}$. Inductively, extend this path to include the first r vertices of T . Call this path $P_{1,r}$ and assume it contains at most $9k(r-1)$ vertices. Since $\kappa(G - P_{1,r}) \geq 9k(t-r+1)$, there will always exist an x_r, x_{r+1} -path avoiding $P_{1,r}$ provided $r \leq t$. Then, Lemma 3.5 implies a shortest x_r, x_{r+1} -path adds at most $9k$ additional vertices. Since $P_{1,t}$ uses at most $9k(t-1)$ vertices, $G - P_{1,t}$ is still connected. Thus, it is possible to find an x_r, x_1 -path avoiding $P_{1,t}$ and Lemma 3.5 implies it adds at most $9k$ additional vertices. \square

4. σ_2 AND SHORT PATHS THROUGH SPECIFIED VERTICES

The theorem below gives sufficient connectivity and σ_2 conditions to ensure that there is a short path in G containing T , given any vertex set T whose cardinality is sublinear. It requires G have sufficiently large order and “short” in this case means a constant multiple of $|T|$.

Theorem 4.1. *Let $k \geq 1$ be an integer. Let G be a k -connected graph on n vertices such that $\sigma_2(G) \geq \frac{2n}{k+2} + f(k)$ where $f(k)$ is a recursively defined function of k and let $T \subseteq V(G)$ such that $|T| \leq g(n) = o(n)$. Then there exists a path in G on at most $O(|T|)$ vertices containing all the vertices of T .*

Proof. Let G and T satisfy the hypotheses of the Theorem and let $t = |T|$. Define a recursive sequence: $b_0 = 10k^2$ and, for $i \geq 1$, $b_i = 10k^2 \left(\frac{k+2}{2}\right)^2 + \sum_{j=0}^{i-1} b_j$. Let $f(k) = 2b_{k+1}$.

Assume G is t -connected. Then Lemma 3.1 implies there exists a cycle containing all the vertices of T , and Lemma 3.4 implies the smallest such cycle contains at most $8kt$ vertices. Thus, there exists a path containing all vertices of T using at most $O(|T|)$ vertices, and the Theorem follows. Thus, we can assume G is k -connected and $k < t$.

Now, we will iteratively find minimum cut sets, denoted by S_i 's. We will collect all the vertices of these iteratively selected cut sets into a set denoted by \mathcal{S} . The connected components of $G - \mathcal{S}$ will be denoted by G_j 's. The indices of the S_i 's and the G_j 's are not important except that we will eventually argue that there are at most $k+2$ of the G_j 's. Thus, we allow arbitrary reordering of the indices as needed. (See Table 1 for an illustration of this algorithm.)

For iteration one, we let S_0 be a minimum cut set of G ; so $|S_0| < t$. Let $\mathcal{S} = S_0$ and G_1, G_2, \dots, G_l be the connected components of $G - \mathcal{S}$. If each G_j is complete or has connectivity at least $30kt$, we stop.

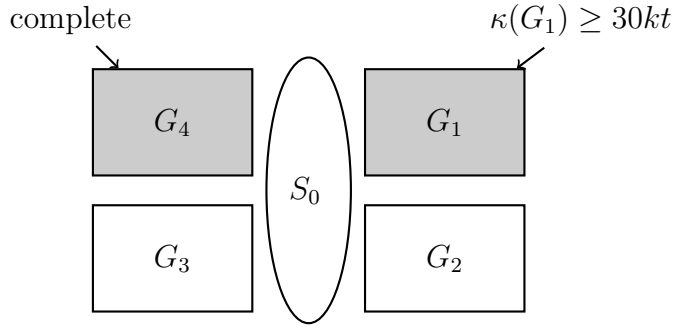
Otherwise, we proceed to iteration two where, for each component G_j in $G - \mathcal{S}$ that is not complete and has connectivity less than $30kt$, we find a minimum cut set S_i . We add all the vertices from these new cut-sets to \mathcal{S} . If each component of $G - \mathcal{S}$ is either complete or has connectivity at least $30kt$, we stop.

In general, on the i th iteration, we stop if all components of $G - \mathcal{S}$ are either complete or have connectivity at least $30kt$.

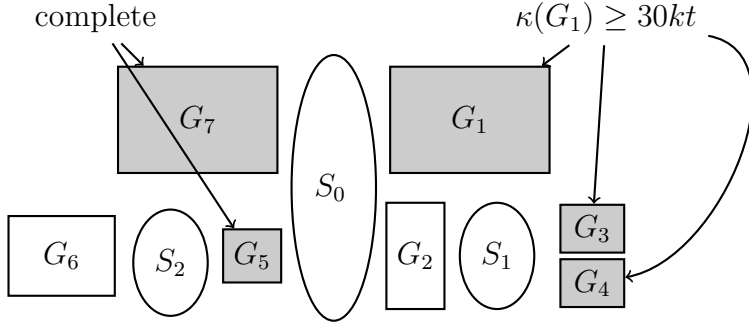
We claim the process stops with at most $k+2$ connected components, G_j . Observe that the first iteration produces at least two components and each subsequent iteration produces at least one additional component. Thus, the i th iteration must end in a collection of cut sets producing at least $i+1$ connected components.

Proceed by contradiction and find the first iteration, i_0 , such that $G - \mathcal{S}$ contains more than $k+2$ connected components. From the previous observation, we know that $i_0 \leq k+2$. In fact, we know that at most $k+2$ cut sets in total were deleted from G . Since, every cut set has order less than $30kt$, at the end of iteration i_0 , $|\mathcal{S}| < (k+2)(30kt)$.

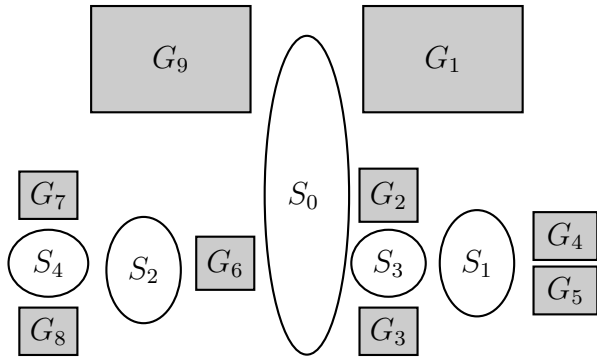
TABLE 1. Cut-Set Selection Algorithm



Iteration 1: A minimum cut set S_0 results in four connected components, G_1 , G_2 , G_3 , and G_4 . So $\mathcal{S} = S_0$ and $k \leq s < t$.



Iteration 2: Minimum cut sets S_1 and S_2 are found in noncomplete components of $G - \mathcal{S}$ with connectivity less than $30kt$. Now, $\mathcal{S} = S_0 \cup S_1 \cup S_2$, $s < t + 2(30kt)$ and $G - \mathcal{S}$ results in seven connected components.



Iteration 3: Minimum cut sets S_3 and S_4 are found in noncomplete components of $G - \mathcal{S}$ with connectivity less than $30kt$. Now, $\mathcal{S} = \cup_{i=0}^4 S_i$, $s < t + 4(30kt)$ and $G - \mathcal{S}$ results in 9 connected components. The algorithm would terminate with all components either complete or with connectivity at least $30kt$.

Let $v_j \in G_j$ for all j . Now, applying the degree sum condition to consecutive pairs of vertices in the connected components, we have

$$(3) \quad \sum_j |G_j| > \frac{1}{2} \sum_j (d(v_j) + d(v_{j+1}) - 2|\mathcal{S}|) \geq \frac{k+3}{2} \left(\frac{2n}{k+2} + f(k) - 2|\mathcal{S}| \right),$$

(where indices of v_j are taken modulo the number of components in $G - \mathcal{S}$.) Thus, using equation 3, we produce the contradiction:

$$(4) \quad n = |\mathcal{S}| + \sum |G_i| > |\mathcal{S}| + \left(\frac{k+3}{k+2} \right) n + \frac{(k+3)f(k)}{2} - (k+3)|\mathcal{S}| > \frac{k+3}{k+2} n - (k+2)^2(30kt) > n,$$

for n sufficiently large and $t = o(n)$. Thus, we can assume that the iterative selection of cut sets terminates in at most $k+1$ iterations and with at most $k+2$ connected components.

Since \mathcal{S} is well-defined, let $s = |\mathcal{S}|$ and note that $s < 30kt * (\text{the number of components} - 1) \leq 30k(k+1)t = o(n)$.

Next, we establish that following claims which will allow the application of Lemmas 3.4, 3.5, and 3.6 to each component G_j .

Claim 1: For every \mathcal{S} and resulting collection of G_j 's, $\sigma_2(G_j) > \frac{2|V(G_j)|}{k+2}$.

Let $L = \{v \in V(G) \mid d(v) < \frac{1}{2}\sigma_2(G)\}$ and $H = V(G) - L$ (where L stands for *low-degree vertices* and H for *high-degree vertices*). Observe that the graph induced by L is complete. Thus, at most one component of $G - \mathcal{S}$ can contain any vertices from L and every non-complete component must contain vertices from H .

If $G - \mathcal{S}$ has at least three components, then for every non-complete component, say G_{j_1} , there exists a different component, say G_{j_2} , containing a vertex, $v \in G_{j_2} \cap H$. Since $N(v) \cap G_{j_1} = \emptyset$, we know $|V(G_{j_1})| \leq n - d(v) \leq n - \frac{\sigma_2(G)}{2}$, or, equivalently, $\frac{\sigma_2(G)}{2} \leq n - |V(G_{j_1})|$.

Now, for n sufficiently large,

$$\begin{aligned}
 \sigma_2(G_{j_1}) &\geq \frac{2n}{k+2} + f(k) - 2s \geq \frac{2|V(G_{j_1})| + 2(n - |V(G_{j_1})|)}{k+2} + f(k) - 2s \\
 (5) \quad &> \frac{2|V(G_{j_1})|}{k+2} + \frac{2(\sigma_2(G)/2)}{k+2} - 2s = \frac{2|V(G_{j_1})|}{k+2} + \frac{2n}{(k+2)^2} - 2s \\
 &> \frac{2|V(G_{j_1})|}{k+2}
 \end{aligned}$$

since $s = o(n)$.

If $G - \mathcal{S}$ contains exactly two components, the argument and arithmetic above applies, unless one of the components, say G_1 , contains only vertices from L . In this case, observe $\mathcal{S} = S_0$ where $|S_0| = s < t$, G_1 is complete, and $|G_1| < |G_2|$.

If $|G_1| \geq (k+2)t$, then $\frac{|G_1|+s}{k+2} > t > s$ and, using $|V(G_1)| + |V(G_2)| + s = n$, we conclude

$$(6) \quad \delta(G_2) \geq \frac{n}{k+2} + \frac{f(k)}{2} - s = \frac{|V(G_2)|}{k+2} + \frac{f(k)}{2} + \frac{|V(G_1)|+s}{k+2} - s > \frac{|G_2|}{k+2}.$$

On the other hand, if $|G_1| < (k+2)t$, then $\delta(G_2) > \sigma_2(G) - (|V(G_1)| + s) - s \geq \frac{2n}{k+2} - (k+4)t > \frac{|V(G_2)|}{k+2}$, since $t = o(n)$. In either case, $\sigma_2(G_2) > 2\frac{|G_2|}{k+2}$, and the claim holds. \square

Claim 2: Either a component G_i of $G - \mathcal{S}$ is complete or $n_i = |V(G_i)| = o(t)$.

If G_i is not complete, then G_i contains a vertex of degree at least $\sigma_2(G)/2$. Now, $n_i > \frac{n}{k+2} - s > \frac{n}{k+2} - 30k(k+1)t$ and, since $t = o(n)$, the claim holds. \square

Before explicitly constructing a path containing T , some additional notation will be introduced. Let $T_{\mathcal{S}} = T \cap \mathcal{S}$. Define T_L , T_H , $T_{L \cap \mathcal{S}}$, and $T_{H \cap \mathcal{S}}$ analogously.

Case 1: Assume $G - \mathcal{S}$ has r connected components where $2 \leq r \leq k$.

First observe that it is sufficient to construct a cycle on $O(|T_H|)$ vertices containing all of T_H because the vertices in T_L form a clique and any shortest path from T_L to the cycle could trivially be extended to a path on at most $O(|T|)$ through all of T . Thus, we show how to construct such a cycle. Second, observe that any component of $G - \mathcal{S}$ of order less than k must be contained in L and thus there is at most one of these. Finally, note that since the number of components is at most k , the number of cut-sets producing \mathcal{S} is at most $k - 1$. Hence, $s \leq 30k(k - 1)t$.

Because the steps used in Case 1 will be essentially the same in Cases 2 and 3, the steps are given names for ease of reference later. To aid the reader, pictures of the various steps are shown in Table 2.

[Construct new graph G' and cycle C' .] Construct a new graph G' by adding a new vertex u_i adjacent to all the vertices of G_i , for every component G_i containing vertices from T_H . Since all such G_i have at least k vertices and at most k new vertices were added, the new graph G' is still k -connected and therefore contains a cycle through all the u_i 's. Choose a shortest such cycle. By Lemma 3.4, this shortest cycle contains at most $k(3k + 5) \leq 8k^2$ vertices and call this cycle C' .

For ease of reference, we will split the cycle, C' into $2r' \leq 2r$ segments: $P'_1, Q'_1, P'_2, Q'_2, \dots, P'_{r'}, Q'_{r'}$ as follows. For each i , P'_i will be the segment of the cycle between two consecutive vertices of \mathcal{S} that contains u_i . That is, $P'_i = w_i, y_1, \dots, u_i, \dots, y_l, z_i$, where $w_i, z_i \in \mathcal{S}$ and $y_1, \dots, y_l \in V(G_i)$. The segment between P'_i and P'_{i+1} will form Q'_i (where indices are taken modulo r .) In particular, Q'_i is a path from z_i to w_{i+1} and, if $z_i = w_{i+1}$, then Q'_i is a single vertex. Since $|V(C')| \leq 8kr' \leq 8k^2$ and each path P'_i contains at least three vertices other than w_i and z_i , we conclude $\cup_i |V(Q'_i)| < 8k^2$.

[Identify T_i 's.] For every $x \in T_{\mathcal{S} \cap H}$, there exists some i such that x has at least $\frac{\sigma_2(G) - 2s}{2k} \geq \frac{n}{k(k+2)} - 30k^2t$ neighbors in G_i . Thus, as $t = o(n)$ and n is sufficiently large, for every such x , it is possible to associate a pair of neighbors in some G_i unique to x . If $x \in T_{\mathcal{S} \cap H} - V(\cup_i Q'_i)$, identify such a pair and label them $v_{x,1}$ and $v_{x,2}$. In this case, we say the vertex x belongs to component G_i .

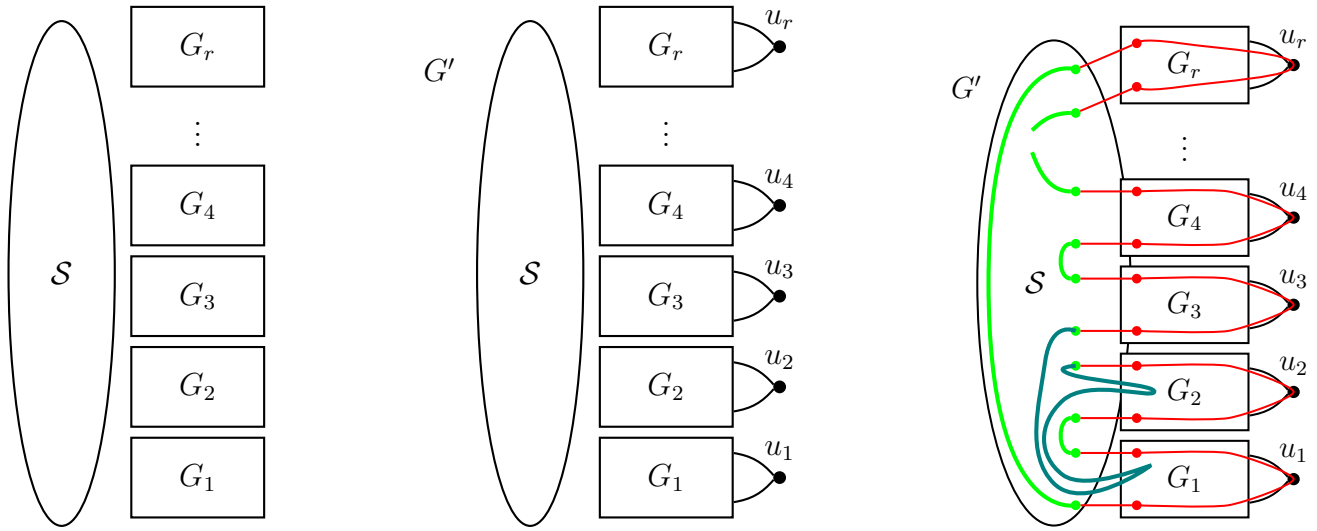
For each component G_i , we define T_i to be the union of all vertices in $(T_H \cap V(G_i)) - (\cup_i Q'_i)$ along with all associated pairs, $v_{x,1}$ and $v_{x,2}$, chosen in G_i . Observe that $|\cup_i T_i| \leq 2t$, and that this upper bound would be achieved only if $T = T_H$, $T \cap (\cup_i Q'_i) = \emptyset$, and $T \subseteq \mathcal{S}$. Trivially, $|T_i| \leq 2t$.

[Replace each P'_i with new paths P_i containing the vertices of T_i in a convenient order and leaving $\cup_i Q'_i$ intact.] Specifically, we want to construct new w_i, z_i -paths, P_i , such that all internal vertices are contained in G_i and vertices of T that belong to G_i , all vertices of T_i lie on the path, and such that any associated pairs ($v_{x,1}$ and $v_{x,2}$) occur on P_i as a $P_3 = v_{x,1}, x, v_{x,2}$.

If G_i is complete, then we trivially construct a w_i, z_i -path using y_i, y_l , and vertices of T_i in any order and $|V(P_i)| \leq 4 + |T_i|$. If G_i is not complete, then we know $\kappa(G_i) \geq 30kt$. Thus, $\kappa(G_i - \cup_i V(Q'_i)) > 30kt - 8k^2 > 0$. Constructing P_i requires finding a path through at most $|T_i| + 2$ vertices where $|T_i| + 2 = o(n_i)$ and $\sigma_2(G_i) \geq 2n_i/(k + 2)$. Thus, Lemma 3.6 implies there exists a path on at most $9k(|T_i| + 2)$ vertices through the specified vertices in any chosen order. Thus, we can always include associated pairs consecutively on a 3-path of the form $v_{x,1}, x, v_{x,2}$.

Observe that $|V(P_i)| \leq 9k(|T_i| + 2) + 2$ and $\cup_i (V(P_i)) \leq 9k(2t + k) + 2k < 29kt$.

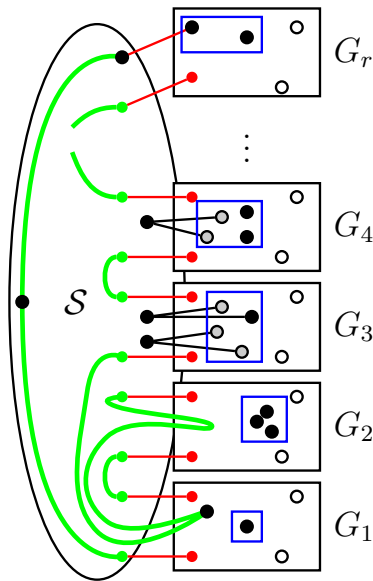
TABLE 2. Cartoon Illustrating the Proof of Theorem 4.1



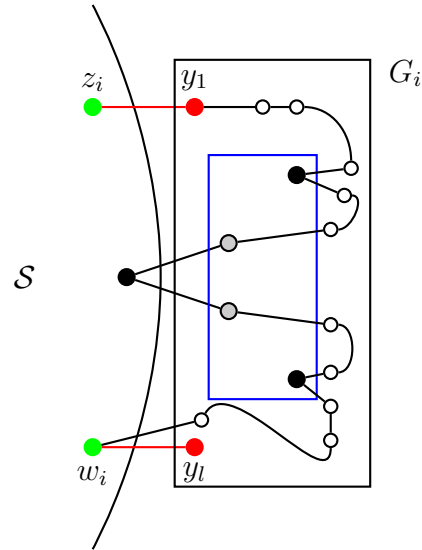
This shows \mathcal{S} and the components of $G - \mathcal{S}$

This shows G' with added vertices u_i .

This shows C' (in red, green, and teal) in G' . Note that the the Q'_i 's are depicted in green (and teal) which is the portion of C' left intact. The teal section (Q'_3) illustrates how complicated these connector sections may be. They may include vertices from G outside \mathcal{S} and possibly vertices from T .



In the diagram above, black vertices are in T , gray vertices are designated neighbors of vertices of $T_{\mathcal{S}}$ not on any green path. For each G_i , the set T_i is in a blue box. Observe that vertices of T on any Q'_j (in green) are not included in any T_i .



This diagram illustrates how the red portion of C' is replaced with a path containing the vertices of T_i . As before, vertices of T are black, associated neighbors of vertices that belong to G_i are in gray. Observe that y_1 and y_l may or may not be on this new path. It is enough to know that z_i and w_i have distinct neighbors in G_i .

[Form a new cycle C containing T_H .] Form a new cycle $C = P_1Q'_1P_2Q'_2 \cdots P_rQ'_r$. We know $|V(C)| \leq |\cup_i V(P_i)| + |\cup_i V(Q'_i)| < 29kt + 8k^2 < 37kt$. Observe that $T_H \subseteq V(C)$, and thus, Case 1 is proved.

Case 2: Assume $G - \mathcal{S}$ has $k + 1 \geq 3$ connected components.

We will begin as in Case 1. As before, construct G' , which may require adding $k + 1$ vertices. Construct C' through u_1, u_2 , and u_k and define P'_i and Q'_i as in Case 1. As before, $|V(C')| \leq 8k^2$ and hence $|\cup_i V(Q'_i)| < 8k^2$.

In this case, we claim that $\delta(G) \geq \frac{k-1}{k(k+1)}n$, and therefore every vertex in \mathcal{S} could be assigned a unique pair of neighbors in G_i (called $v_{1,x}, v_{2,x}$).

Let x be a vertex of degree at least $\sigma_2(G) - \delta(G)$. If there exists a choice of x in the smallest or second smallest component of $G - \mathcal{S}$, then $\sigma_2(G) - \delta(G) \leq d(x) < \frac{n-s-1}{k} + s$. Hence, $\delta(G) > \frac{k-1}{k(k+1)}n$ and the claim holds. If no such vertex x exists, then every vertex in these smallest two components must be adjacent to every vertex of minimum degree. This implies that the number of vertices in these two components is at most $\delta(G)$. Let y_1 and y_2 be vertices from each of these components. Now, $\sigma_2(G) = \frac{2n}{k+1} + f(k) \leq d(y_1) + d(y_2) < \delta(G) + 2s$. Thus, $\delta(G) > \frac{2n}{k+1} + f(k) - 2s > \frac{k-1}{k(k+1)}n$, since $s \leq 30k^2t$ and $t = o(n)$.

So, unlike Case 1, for every $x \in T_{\mathcal{S}}$ (not just those in $T_{S \cap H}$), there exists some G_i such that x can be assigned a unique pair of neighbors in G_i (called $v_{1,x}, v_{2,x}$) and we say x belongs to G_i . Thus, in Case 2, if $x \in T_{\mathcal{S}} - \cup_i V(Q'_i)$, we associate such a pair.

Similar to Case 1, for every $i \in \{1, 2, \dots, k+1\}$, a set T_i will be identified, but in this case it will not be restricted to T_H . Specifically, T_i is the union of the vertices in $(T \cap V(G_i)) - (\cup_i Q'_i)$ along with all associated pairs, $v_{x,1}$ and $v_{x,2}$, chosen in G_i . Now every vertex in T either lies on some Q'_i , is contained in some T_i , or has a pair of neighbors in some T_i . Let $t_{k+1} = |T_{k+1}|$ and let $t_0 = \cup_{i=1}^k |T_i|$.

By replacing each P'_i with a P_i as in Case 1, we form a cycle C containing all vertices of T except those in $G_{k+1} - \cup_i V(Q'_i)$. As before, $|V(C)| \leq |\cup_i V(P_i)| + |\cup_i V(Q'_i)| < 29kt_0 + 8k^2 < 37kt_0$.

Observe that $t_{k+1} \leq 2t$ and, either G_{k+1} is complete or $\kappa(G_{k+1} - \cup_i Q'_i) > 30kt - 8k^2 > 9k(2t)$. Thus, $G_{k+1} - \cup_i Q'_i$ contains a cycle through all of T_{k+1} in any order. Let C_{k+1} be a smallest such cycle and observe that $|V(C_{k+1})| \leq 9kt_{k+1}$.

Since G is connected, there must exist a path from C to C_{k+1} . Now these two cycles and a shortest path between them contains at most $29kt_0 + 8k^2 + 9kt_{k+1} + 9k \leq 38kt$ vertices and trivially contains a path through all of T . Thus, the theorem holds for Case 2.

Case 3: Assume $G - \mathcal{S}$ has $k + 2$ connected components.

As in Case 2, we are guaranteed that every vertex $v \in T_{\mathcal{S}}$ can be assigned a unique pair of neighbors in some G_i . Furthermore, in this case, no component of $G - \mathcal{S}$ can have order less than k .

Claim 2: There exists some pair of the connected components, say G_{k+1} and G_{k+2} , and some cut set, S_i , such that there exist $10k^2$ vertex disjoint P_3 's from G_{k+1} to G_{k+2} through S_i .

We will proceed by contradiction and let S_0, S_1, \dots, S_r be the cut sets in the order in which they were chosen and such that $G - \cup_i S_i$ has connected components G_1, G_2, \dots, G_{k+2} .

Recall the definition of the recurrence relation: $b_0 = 10k^2$ and $b_i = 10k^2 \left(\frac{k+2}{2}\right)^2 + \sum_{j=0}^{i-1} b_j$.

If $|S_r| \geq b_0 = 10k^2$, any two children of S_r will suffice, and Claim 2 holds.

So, assume $|S_r| < 10k^2$. Furthermore, assume there exists an $i_0 < r$, such that for all $i > i_0$, $|S_i| < b_{r-i}$ but $|S_{i_0}| \geq b_{r-i_0} = 10k^2 \left(\frac{k+2}{2}\right)^2 + \sum_{i=0}^{i_0} b_i$.

Observe that, in the matchings between S_{i_0} its children, we can delete from S_{i_0} all vertices on matching edges to $\{S_{i_0+1}, \dots, S_r\}$ and still have $10k^2 \left(\frac{k+2}{2}\right)^2$ matching edges left. Now all remaining matching edges must go to vertices in $\cup G_j$. As there less than $\left(\frac{k+2}{2}\right)^2$ different pairs of G_j 's in all, at least one pair of children of S_{i_0} has at least $10k^2$ P_3 's through S_{i_0} .

On the other hand, if no such S_{i_0} exists (that is, if $|S_i| < b_{r-i}$ for all i), then $s < f(k)/2$, by definition of $f(k)$. Yet, by considering two vertices v and w in the two smallest components of $G - \{S_1, S_2, \dots, S_r\}$, we observe that

$$\frac{2n}{k+2} + f(k) \leq d(v) + d(w) \leq 2 \left(\frac{n-s}{k+2} + s - 1 \right) < \frac{2n}{k+2} + 2s$$

which implies $s > f(k)/2$. So Claim 2 holds.

Label the cut sets S_i and components of $G - \cup S_i$ such that G_{k+2} and G_{k+1} have at least $10k^2$ P_3 's through a cut set S_r . Now, we repeat the arguments from Cases 1 and 2. Specifically, construct G' by adding as many as $k+2$ new vertices. Construct cycle C' through u_1, u_2, \dots, u_k using at most $8k^2$ vertices. Define subpaths P'_i and Q'_i as before.

Observe that since cycle C uses at most $8k^2$ vertices, there still exist at least $10k^2 - 8k^2 = 2k^2 \geq 2$ paths on three vertices between G_{k+1} and G_{k+2} through S_r . Label these two paths Q_{k+1} and Q_{k+2} and call the middle vertices on these paths m_1 and m_2 .

For every $x \in (T \cap \mathcal{S}) - (\cup_i Q'_i) - \{m_1, m_2\}$, associate a unique pair of neighbors in some G_i , called $v_{1,x}$ and $v_{2,x}$. For each $i \in \{1, 2, \dots, k+2\}$, identify T_i as in Case 2. Let $t_{k+2} = |T_{k+2}|$, $t_{k+1} = |T_{k+1}|$, and $t_0 = \cup_{i=1}^k |T_i|$. Thus, $t_0 + t_{k+1} + t_{k+2} \leq 2t$. Replace each P'_i with a P_i as in previous cases to form the cycle C on at most $29kt_0 + 8k^2 < 37kt_0$ vertices.

Finally, construct path P_{k+1} from m_1 to m_2 in G_{k+1} such that all internal vertices are contained in $V(G_{k+1})$ and vertices of T that belong to G_i , all vertices of T_i lie on the path, and such that any associated pairs $(v_{x,1}$ and $v_{x,2})$ occur on P_{k+1} as a $P_3 = v_{x,1}, x, v_{x,2}$. Find the analogous path P_{k+2} in G_{k+2} and vertices that belong to G_{k+2} . Together $P_{k+1}Q_{k+1}P_{k+2}Q_{k+2}$ form a cycle, C_1 on at most $9k(t_{k+1} + t_{k+2} + 4) + 2$ vertices.

Any shortest path between C and C_1 contains our desired path and uses at most:

$$|V(C)| + |V(C_1)| + 9k \leq 29kt_0 + 8k^2 + 9k(t_{k+1} + t_{k+2} + 4) + 2 + 9k < 38kt$$

concluding Case 3.

Thus, in all three cases, a path on at most $O(t)$ vertices can be found through the set T , and the Theorem holds. \square

Corollary 4.1. *Let $k \geq 1$ be an integer. Let G be a k -connected graph on n vertices such that $\sigma_2(G) \geq \frac{2n}{k+2} + f(k)$ where $f(k)$ is a recursively defined function of k . Then there exists a vertex dominating path in G on at most $O(\ln n)$ vertices.*

Proof. Any G satisfying the hypotheses of the Corollary contains a dominating set of vertices of order at most $O(\ln n)$ using Lemma 3.3. \square

5. CONCLUDING REMARKS

Observe that Example 2.1 shows that Corollary 4.1 is close to sharp. That is, the example has no dominating path and $\sigma_2(G) = \frac{2n-4(k+1)}{k+2}$.

We conjecture that this is the extremal example.

Conjecture 5.1. *Let $k \geq 1$ be an integer. Let G be a k -connected graph on n vertices such that $\sigma_2(G) > \frac{2n-4(k+1)}{k+2}$. Then there exists a vertex dominating path in G on at most $O(\ln n)$ vertices.*

In fact, using the same techniques as in Theorem 4.1, it is possible to prove the cycle analogues of Theorem 4.1 and Corollary 4.1, stated below.

Theorem 5.1. *If G is k -connected and $\sigma_2(G) \geq \frac{2n}{k+1} + f(k)$ and $T \subseteq V(G)$ such that $|T| = o(n)$, then there exists a cycle through T on at most $O(|T|)$ vertices.*

Corollary 5.1. *Let $k \geq 1$ be an integer. Let G be a k -connected graph on n vertices such that $\sigma_2(G) \geq \frac{2n}{k+1} + f(k)$ where $f(k)$ is a recursively defined function of k . Then there exists a vertex dominating cycle in G on at most $O(\ln n)$ vertices.*

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