INTERSECTING DOMINO TILINGS

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Abstract. In this note we consider an Erdős-Ko-Rado analog of tilings. Namely given two tilings of a common region we say that they intersect if they have at least one tile in the same location. We show that for a domino tiling of the $2\times n$ strip that the largest collection of tilings which pairwise intersect are counted by the Fibonacci numbers. We also solve the problem for tilings of the $3\times(2n)$ strip using dominos.

1. Introduction

A well known problem in set theory is to construct a large family of k-element subsets of $\{1,\ldots,n\}$ so that any two of the sets have at least ℓ elements in common. One easy construction is to take all the k-element sets containing $\{1,\ldots,\ell\}$, of which there are $\binom{n-\ell}{k-\ell}$. A famous result in extremal set theory says that this is essentially the best possible, or in other words you cannot be more clever than doing the obvious thing.

Theorem 1.1 (Erdős-Ko-Rado [5]). *Let $\mathcal{F}$ be a family of k-element subsets of $\{1,\ldots,n\}$ any two of which have at least ℓ elements in common. Then for $n\geq(k-\ell+1)(\ell+1)$*

$$|\mathcal{F}|\leq\binom{n-\ell}{k-\ell}.$$

Similar problems can be looked at for any combinatorial object which has a notion of intersection. Objects that have previously been studied include permutations [4], set partitions [11], colored sets [2], arithmetic progressions [6], strings [7], and vector spaces [8]. We want to consider a new intersection problem, namely the intersection of tilings.

A tiling consists of covering a region using tile pieces from some given set so that the region is completely covered without overlaps (more about tilings can be found in the survey paper by Ardila and Stanley [1]). We say that two tilings of the region intersect if there is a tile placed in the same position on both tilings. For example, Figure 1 shows two tilings of a 4×5 board using dominoes (pieces of size 1×2). The shaded tile is placed in the same position in both tilings so these tilings intersect.

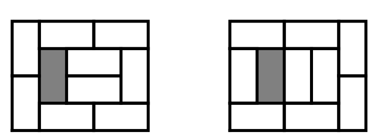


Figure 1. An example of intersecting tilings.

We will find the maximal size of families of intersecting tilings for the cases of tiling the $2\times n$ strip and the $3\times(2n)$ strip by using dominoes.

2. Tilings of $2\times n$ using dominoes

A classic characterization of the Fibonacci numbers is that it counts the number of tilings of the $2\times n$ strip using dominoes, namely there are F_{n+1} such tilings.

If we wanted to form a large intersecting family, then one obvious family to try is to take all tilings that start with a vertical domino, this gives a family with F_n intersecting tilings. We now want to show that this is the best possible (this is always the hardest part of an Erdős-Ko-Rado problem).

Theorem 2.1. *Let $\mathcal{T}$ be an intersecting family of tilings of the $2\times n$ strip using dominoes. Then $|\mathcal{T}| \le F_n$.*

Proof. Consider the graph which is formed by taking all possible tilings and putting an edge between two tilings it they do **not** intersect. The problem of finding a maximal intersecting family is equivalent to finding a maximal independent set in this graph. We can split the vertices into two sets $\mathcal{H}$ and $\mathcal{V}$. Where $\mathcal{H}$ is the F_{n-1} tilings that start with two horizontal tiles and $\mathcal{V}$ is the F_n tilings that start with a vertical tile. By definition, all edges in the graph are between $\mathcal{H}$ and $\mathcal{V}$ (i.e., the graph is bipartite).

We claim that there is a matching between $\mathcal{H}$ and a subset of $\mathcal{V}$. To see this, suppose that we have a tiling T in $\mathcal{H}$. Then we can decompose this tiling into a sequence of blocks where a block consists of two horizontal tiles followed by any number of vertical tiles. We now map $T \to S$ block by block using the rule shown in Figure 2. For any $T \in \mathcal{H}$ the resulting S will start with a vertical tile and so is in

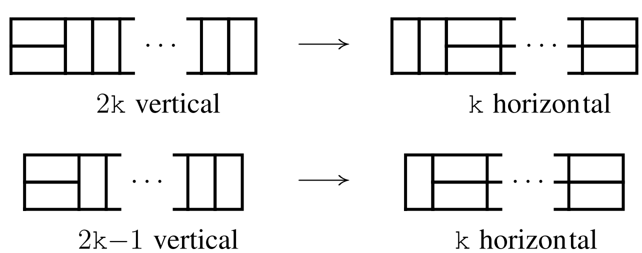


Figure 2. The rule for forming the matching between $\mathcal{H}$ and $\mathcal{V}$.

$\mathcal{V}$, further block by block it can be seen that S and T have no common tile, so there is an edge between S and T. Finally it is easy to check that this map is 1-to-1, so gives our desired matching.

Since there is a matching from every element of $\mathcal{H}$ to an element of $\mathcal{V}$ it follows that for any subset $\mathcal{Q}$ of $\mathcal{H}$ that the number of elements in $\mathcal{V}$ adjacent to $\mathcal{Q}$ has size at least $|\mathcal{Q}|$. (This is the rarely used direction of Hall's Marriage Theorem.) Now suppose that $\mathcal{T}$ is an intersecting family and let $\mathcal{Q} = \mathcal{T} \cap \mathcal{H}$ and $\mathcal{R} = \mathcal{T} \cap \mathcal{V}$. Since the elements of $\mathcal{R}$ cannot be adjacent to elements of $\mathcal{Q}$ the above comment implies that $|\mathcal{R}| \le |\mathcal{V}| - |\mathcal{Q}|$. So we have

$$|\mathcal{T}| = |\mathcal{Q}| + |\mathcal{R}| \le |\mathcal{Q}| + (|\mathcal{V}| - |\mathcal{Q}|) = |\mathcal{V}| = F_n. \qquad \square$$

A natural extension is to consider the same problem but where we now require at least ℓ intersecting tiles. A first guess is that it will have size $F_{n+1-\ell}$, i.e., by taking ℓ vertical dominoes at the start and then an arbitrary tiling of the remaining $2\times(n-\ell)$ strip. However, this in not true in general. For example for $n = 6$ and $\ell = 2$ there is a family of $6 > 5 = F_5$ tilings of 2×6 that pairwise intersect in at least 2 or more tiles, this is shown in Figure 3.

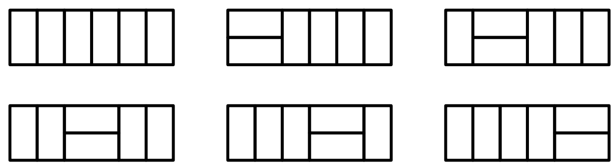


FIGURE 3. A 2-intersecting family of tilings for 2×6.

More generally for $\ell \geq 3$ we can construct a set of $\ell+4$ prefixes of length $\ell+4$ by taking all domino tilings of length $\ell+4$ with at most one horizontal pair of dominoes (the case $\ell = 2$ is illustrated in Figure 3). Note that these prefixes pairwise intersect in at least ℓ places and so if we take all domino tilings of $2\times n$ that start with these prefixes that gives us a family of size $(\ell + 4)F_{n+1-\ell-4}$ which intersect in at least ℓ places. But since

$$\ell + 4 \geq 7 > 6.854101954\cdots = \left(\frac{1+\sqrt{5}}{2}\right)^4 = \phi^4$$

then for n large this is larger than $F_{n+1-\ell} \approx \phi^4 F_{n+1-\ell-4}$.

3. Tilings of $3\times(2n)$ using dominoes

We now turn to tilings of the $3\times(2n)$ board. We first count the number of such tilings (this has been done previously and is `A001835` in the `OEIS` [12]; also see [10]). A commonly used approach is to set up a system of linear recurrences and then solve the system, we will do a variation where we count the number of weighted walks in a small graph.

The basic idea is to break the $3\times(2n)$ strip into n small blocks of size 3×2, and consider how horizontal dominoes can intersect the column between consecutive blocks. Since the area of each block is even, it follows that in the columns we must have an even number of horizontal dominoes. This gives the four possibilities shown in Figure 4, the fourth of which cannot happen in a tiling of $3\times(2n)$, we will refer to the remaining possibilities, from left to right, as 工, 百 and 旦.

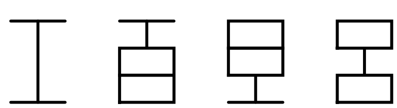


FIGURE 4. The different configuration of horizontal dominoes between blocks.

To count the total number of tilings we can take all possible configurations of horizontal dominoes in the columns and then count the ways to fill in the remaining untiled portion of the strip. We can do this by using weighted walks in a small directed graph where the vertex set is {, , } and the weight of an edge is the number of ways to fill in the unused area of a block between the two columns. For instance there are 3 ways to fill in a $3{\times}2$ strip so there is a loop of weight 3 for the edge . Similarly, edges , , , ,  and  have weight 1 since there is only one way to fill in the block between the two columns, while  and  have weight 0 since there is no way to fill in the uncovered area using dominoes. This gives us the following adjacency matrix for the graph.

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Since the left and right sides of the $3{\times}(2n)$ board correspond to  we need to find the sum of the weight of walks of length n in the graph that start and end at . This is equivalent to finding the $(1,1)$ entry of A^n. The eigenvalues of A are $2+\sqrt{3}$, $2-\sqrt{3}$ and 1, using these along with their eigenvectors to form projection matrices we have

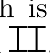
$$A^n = \big(2+\sqrt{3}\,\big)^n \begin{pmatrix} \frac{3+\sqrt{3}}{6} & \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} & \frac{3-\sqrt{3}}{12} & \frac{3-\sqrt{3}}{12} \\ \frac{\sqrt{3}}{6} & \frac{3-\sqrt{3}}{12} & \frac{3-\sqrt{3}}{12} \end{pmatrix}$$
$$+ \big(2-\sqrt{3}\,\big)^n \begin{pmatrix} \frac{3-\sqrt{3}}{6} & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{6} & \frac{3+\sqrt{3}}{12} & \frac{3+\sqrt{3}}{12} \\ -\frac{\sqrt{3}}{6} & \frac{3+\sqrt{3}}{12} & \frac{3+\sqrt{3}}{12} \end{pmatrix} + 1^n \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Taking the sum of the $(1,1)$ entries we have established the following.

Proposition 3.1. *The number of domino tilings of* $3{\times}(2n)$ *is*

$$T_n = \frac{3+\sqrt{3}}{6}\big(2+\sqrt{3}\,\big)^n + \frac{3-\sqrt{3}}{6}\big(2-\sqrt{3}\,\big)^n.$$

Looking at the possible forms of the $3{\times}2$ blocks we get nine possible shapes (nine is also the sum of the entries of A). These are shown in Figure 5. The tiles split into three groups, "blue" tiles with a single horizontal domino on the top, "red" tiles with a single horizontal domino on the bottom and a "universal" tile. Since *every* $3{\times}2$ block has at least one horizontal domino then any $3{\times}(2n)$ tiling which uses a universal tile will intersect every other tiling, i.e., it is universally intersecting. It turns out that these are the only universally intersecting tilings.

We now count the number of tilings that do not have a universal tile. The previous approach is easily adopted and the only change is to remove a single possibility between , namely the one with three horizontal dominoes. This gives the following matrix:

$$B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

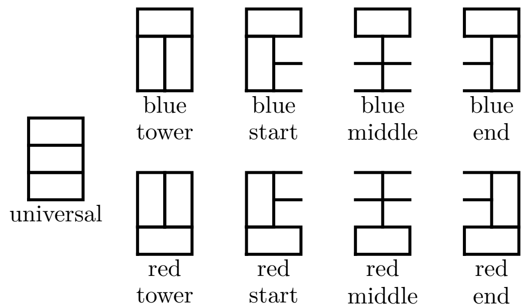


FIGURE 5. The possible 3×2 blocks.

This matrix has eigenvalues 3, 1 and 0, so that for $n \geq 1$ we have, similarly to before,

$$B^n = 3^n \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \end{pmatrix} + 1^n \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Taking the sum of the $(1,1)$ entries we have the following.

Proposition 3.2. *The number of domino tilings of* $3\times(2n)$ *without the universal tile is* $S_n = 2{\cdot}3^{n-1}$.

Given the nice form for this answer we might expect a simple explanation. One way to do this is to note that for tilings of 3×2 without the universal tile there are only two possibilities, i.e., either a red tower or a blue tower. Then we note that given a tiling of $3\times(2n)$ there are three ways to extend it to a tiling of $3\times\big(2(n+1)\big)$, i.e., either add a red tower, add a blue tower, or extend the last run (this is shown in Figure 6 in the case that we end with a red tile), so that $S_{n+1} = 3S_n$.

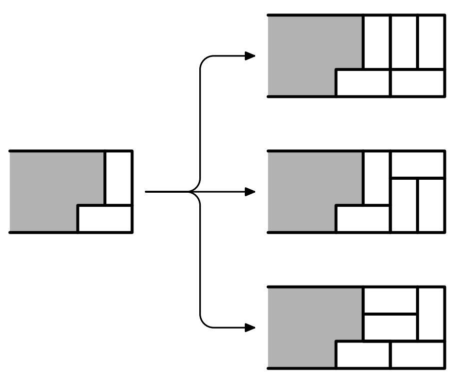


FIGURE 6. The three ways to extend a $3\times(2n)$ tiling.

We are now ready to bound the size of a maximal intersecting family.

Theorem 3.3. *Let* $\mathcal{T}$ *be an intersecting family of tilings of the* $3\times(2n)$ *strip using dominoes. Then*

$$|\mathcal{T}| \leq \frac{3+\sqrt{3}}{6}\big(2+\sqrt{3}\,\big)^n + \frac{3-\sqrt{3}}{6}\big(2-\sqrt{3}\,\big)^n - 3^{n-1}.$$

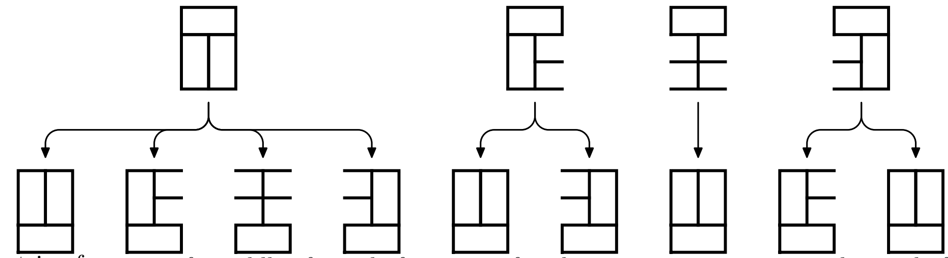


FIGURE 7. Rule for mapping in the $3\times(2n)$ case.

Proof. As in the $2\times n$ case we form a graph where each vertex is a tile and two vertices are connected if they do not intersect. Any tiling which contains the universal tile will be an isolated vertex. The remaining tiles can be split into two groups, those that start with a red tile and those that start with a blue tile. As before this is a bipartition of our graph.

Claim 3.4. *There is a perfect matching in the set of tilings which do not contain the universal tile.*

Before we prove the claim let us show how this will give the statement of the theorem. In an intersecting family we can take any number of the isolated vertices and at most one of the tilings in each edge of the perfect matching. There are $T_n - S_n$ isolated vertices and $\frac{1}{2}S_n$ edges in the perfect matching; it follows that an intersecting family has at most $T_n - \frac{1}{2}S_n$ edges. Now using the results from Propositions 3.1 and 3.2 the result will follow.

To prove the claim we give a mapping between tilings that start with a blue tile to tilings that start with a red tile. So let T be a tiling. We break T into (maximal) blue and red strips. It suffices to give a mapping that takes a blue strip into a red strip of the same size (and vice-versa) so that the tiling and the image does not intersect. Such a mapping is given in Figure 7. It is easy to check that this takes a blue strip into a red strip (or vice-versa) so that the image does not intersect the original tiling. It is also easy to check that this map is injective, since applying the map twice gives us back the original tiling. This concludes the proof. □

4. Concluding remarks

Tiling problems have been very popular (both in looking at existence and enumeration of tilings). Looking for maximal intersecting family of tilings opens up a new avenue of investigation of tilings. We have restricted ourselves to domino tilings of the $2\times n$ and $3\times(2n)$ boards where we look only at intersection. One could look at the more general problem where we look for intersection in at least ℓ tiles and also similar problems for domino tilings of $k\times n$ boards.

Besides looking at domino tilings one can consider tilings with squares and dominoes, or squares and "L"s [3], or tetris pieces, or polyominoes (see Golomb's [9] excellent book on the subject which also deals extensively with tiling problems), or hexagonal animals, or three-dimensional tilings. For each problem one can also consider a variety of different board configurations. The possibilities are limited only by the imagination.

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