3-connected $\{K_{1,3}, P_9\}$-free graphs are hamiltonian connected

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Abstract. We investigate pairs of forbidden subgraphs that imply a 3-connected graph is hamiltonian connected. In particular we show that the pair $\{K_{1,3}, P_9\}$ is such a pair. As it is known that P_{10} cannot replace P_9, this result is best possible. Further, we show that certain other graphs are not possible.

Key words. Hamilton-connected, 3-connected, claw-free, dominating, closure

1. Introduction

A graph G on $n \geq 3$ vertices is called *hamiltonian* if G contains a cycle of length n and is called *hamiltonian connected* if every pair of vertices can be connected by a path on n vertices. G is said to be $\{H_1, H_2, ..., H_k\}$-free if G contains no induced subgraph isomorphic to any of the graphs H_i, $i = 1, 2, ..., k$. In this case the graphs $H_1, H_2, ..., H_k$ are said to be forbidden subgraphs for G.

We now establish some notation that we will use throughout the paper. Of particular importance is the claw, $K_{1,3}$, which is the complete bipartite graph with partite sets of size one and three. Also, we will denote the path on k vertices as P_k, and the generalized net as $N(i, j, k)$, which is a triangle with disjoint paths of length i, j, and k, attached to distinct vertices of the triangle. We use $\Gamma(v)$ to denote the neighborhood of v, $\Gamma[v]$ to denote the closed neighborhood of v (i.e. $\Gamma(v) \cup \{v\}$), and $Ł_k$ to denote two triangles connected by a path with k edges. For terms not found here, see [3].

It is well known that the only single forbidden graph which implies G is hamiltonian is P_3. Since both pancyclicity and hamiltonian connectedness imply hamiltonicity, no single forbidden graph (with the exception of P_3) can suffice for these properties either.

Therefore, the interesting question is to classify all pairs of graphs $\{H_1, H_2\}$ such that any graph G which is free of this pair necessarily has the desired hamiltonian property. A complete classification of forbidden pairs that imply a 2-connected graph is hamiltonian was determined by Bedrossian [1] for all graphs and further generalized by Faudree and Gould [7] for all sufficiently large graphs.

A graph G is said to be *pancyclic* if G contains at least one cycle of each length from 3 to $|V(G)|$ and *panconnected* if any two vertices of G are joined by paths of all possible lengths from $dist\{x, y\}$ (the distance between x and y) to $|V(G)|-1$. A complete classification of forbidden pairs which imply a 2-connected graph G is pancyclic has also been determined by Faudree and Gould [7], as well as a classification of forbidden pairs which imply a 3-connected graph G is panconnected.

The next natural question to consider is which forbidden pairs imply that a 3-connected graph G is hamiltonian connected. These pairs have been much harder to determine, and thus a complete classification is not yet known. However, it is known that one of the two forbidden subgraphs must be the claw. In [7], Faudree and Gould began to classify the properties of graphs H_2 that could form a forbidden pair with the claw, $H_1 = K_{1,3}$, implying a 3-connected graph is hamiltonian connected. A strengthening of these properties was given in [2]. These results greatly reduce the number of potential graphs that can play the role of H_2.

We now summarize the known graphs H_2 which, together with the claw, imply that a 3-connected graph is hamiltonian connected. Shepherd [10] has shown that $H_2 = N(1,1,1)$ is one such graph. Faudree and Gould [7] show that $H_2 = N(2,0,0)$ works as well. Chen and Gould [4], added three additional choices for H_2 to the collection of forbidden pairs that imply a 3-connected graph is hamiltonian connected, namely $N(3,0,0)$, $N(2,1,0)$ and P_6. Broersma et. al. in [2] showed that $Ł_1$ also yields the desired result. Recently, in [5], it was shown that the pair $\{K_{1,3}, P_8\}$ implies a 3-connected graph is hamiltonian connected, as well as several generalized nets.

Combining these results, the only remaining possibilities for the graph H_2 are as follows:
(a) P_9;
(b) $N(i, j, k)$ with some restrictions on how large i, j, k can be;
(c) $Ł_k$ with $k \geq 2$;
(d) $Ł_k$ with $k \geq 2$ with tree components attached to either of the two triangles.

In this paper we work to further classify the pairs of graphs such that G being 3-connected and $\{K_{1,3}, H_2\}$-free implies G is hamiltonian connected. The results of this paper are presented in two parts. First, we present infinite families of graphs that are 3-connected and not hamiltonian connected in order to further reduce the list of possible forbidden pairs. In particular, we eliminate $Ł_k$ for $k \geq 7$ as well as $Ł_k$ with k even. Second, we prove the following theorem:

Theorem 1. *Every 3-connected $\{K_{1,3}, P_9\}$-free graph is hamiltonian connected.*

As it is known that there are $\{K_{1,3}, P_{10}\}$-free graphs which are not hamiltonian connected, this result is the best possible.

1.1. Families of 3-connected non-hamiltonian connected graphs

In this section, we further reduce the list of possible graphs H_2 such that every $\{K_{1,3}, H_2\}$-free graph is hamiltonian connected by presenting some infinite families of claw-free graphs that are $\{K_{1,3}, H_2\}$-free, but not hamiltonian connected.

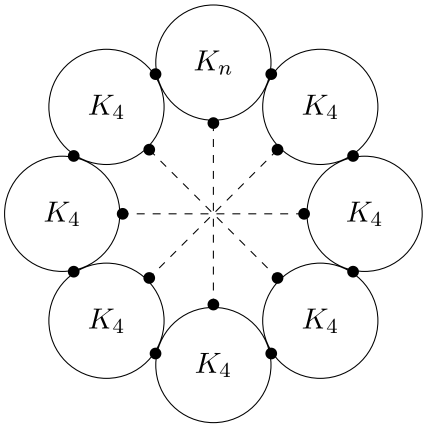


Fig. 1. Non-hamiltonian connected graph which is $\{K_{1,3}, Ł_k\}$-free, $k \geq 7$.

Theorem 2. *If G being 3-connected, claw-free and $Ł_k$-free implies G is hamiltonian connected, then $k \leq 6$*

Proof. Consider the graph G shown in Figure 1. Note that the dashed lines in the figure represent identifications. Since G is 3-connected, claw-free and not hamiltonian connected, it follows that any graph Y such that $\{K_{1,3}, Y\}$-free implies hamiltonian connectedness must be an induced subgraph of G. The two end triangles of the $Ł_k$ must be in two separate cliques. Without loss of generality, we can assume that one triangle is in the copy of K_n. It can also be noted that any two edges between the end triangles must be in separate cliques, otherwise another triangle would be induced. This gives the result that $k \leq 6$ since there are only six remaining cliques in which to place an edge of the $Ł_k$.

Theorem 3. *If G being 3-connected, claw-free and $Ł_k$-free implies G is hamiltonian connected, then k must be odd.*

Proof. Consider the two graphs G_1 and H_1 shown in Figure 2. Since both G_1 and H_1 are claw-free and not hamiltonian connected, it follows that any graph Y, such that $\{K_{1,3}, Y\}$-free implies hamiltonian connectedness, must be an induced subgraph of G_1 and H_1.

Observe that in the graph G_1 there are no triangles that share a common vertex. Thus the shortest distance between two triangles is one. Now choose any triangle T in G_1. To induce a subgraph that connects T with a second triangle via a path of length longer than one, one must include the edge between T and one of the triangles at distance one from T, call this triangle T_1. If we include more than two vertices of T_1, we induce an $Ł_1$. Thus,

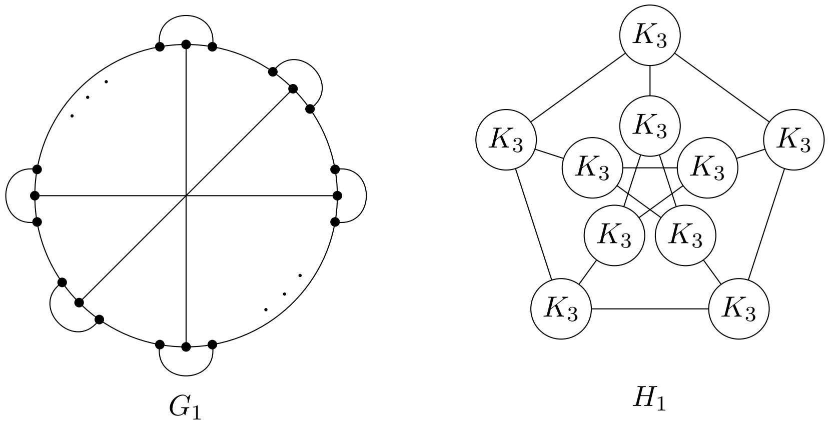


Fig. 2. Non-hamiltonian connected graphs G_1 and H_1 which are $\{K_{1,3}, Ł_k\}$-free for k even.

we can only include one edge of T_1. There is no other triangle at the end of this path of length two, thus we must add in an edge between T_1 and a triangle of distance one from T_1. Continuing in this manner, it can be seen that G_1 contains no induced subgraph $Ł_k$ with k even.

A similar argument shows that H_1 also contains no induced copies of $Ł_k$ with k even.

Combining the results of Theorems 2 and 3, we deduce that the only possible values of k such that a graph G being 3-connected and $\{K_{1,3}, Ł_k\}$-free implies that G is hamiltonian connected are $k = 1, 3, 5$. Note that the $k = 1$ case was proven in [2], thus the only remaining k for which the question is unknown are $k = 3$ and $k = 5$.

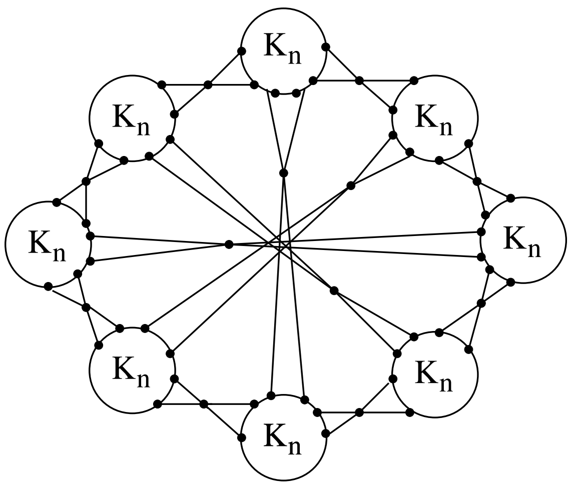


Fig. 3. Graph G_3 eliminating L_3 with trees off the triangles.

Theorem 4. *If G being 3-connected, claw-free and Y-free implies G is hamiltonian connected, then Y cannot be an $Ł_k$ with tree components on either of the two triangles.*

Proof. As before, it is only necessary to provide a class of 3-connected, claw-free graphs that are not hamiltonian connected and do not contain any $Ł_k$ with trees attached to either

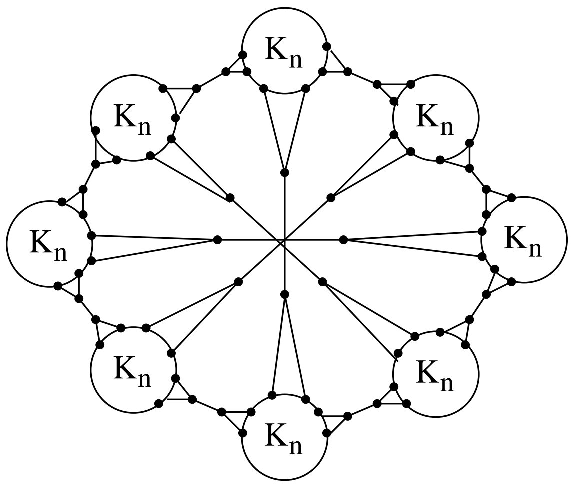


Fig. 4. Graph G_{15} eliminating L_1 and L_5 with trees off the triangles.

of the end triangles as induced subgraphs. By Theorems 2 and 3, it is only necessary to consider Ł$_1$, Ł$_3$, and Ł$_5$ with tree components.

The graph G_{15} in Figure 4 contains no Ł$_1$ with trees attached to either of the triangles and no Ł$_5$ with trees attached to either of the triangles. This can be seen by noting that the only induced Ł$_1$'s and Ł$_5$'s occur with triangles contained in separate cliques. Adding any edge off of either triangle would induce an additional edge between the new vertex and another vertex of the triangle, also in the clique.

Likewise, the graph G_3 in Figure 3 contains no Ł$_3$'s with trees attached to either of the triangles, since any induced Ł$_3$ must occur with the triangles between the cliques.

Theorem 5. *If G being 3-connected claw-free and $N(i, j, k)$-free implies G is hamiltonian connected, then $i + j + k \leq 7$.*

Proof. Consider the graph in Figure 1 used in the proof of Theorem 2. The triangle at the center of the net must be contained in one of the cliques. Without loss of generality it can be assumed to be in the K_n. Any other clique can contain at most one edge from one of the paths off of the triangle, giving a total of at most 7 edges among the three paths.

2. Proof of main result

In this section we focus on proving Theorem 1:

Theorem 1. *Every 3-connected, $\{K_{1,3}, P_9\}$-free graph G is hamiltonian connected.*

2.1. Setup

Suppose G is a $\{K_{1,3}, P_9\}$-free graph. Fix an arbitrary pair of vertices u and v. We wish to show that there is a hamiltonian path between u and v. A tool which we will use to find this hamiltonian path is a closure concept introduced by Kelmans [8]. We denote this closure as $kl_{u,v}$.

Let N_x denote $V(\Gamma[x] \setminus \{u,v\})$. Kelmans defines N_x to be a $kl_{u,v}$-*cone* if the following hold:
(i) $x \in V(G \setminus \{u,v\})$,
(ii) $\Gamma(x) \setminus \{u,v\}$ is connected, and if $\{u,v\} \subseteq V(\Gamma(x))$ then both u and v are adjacent to some vertices in $\Gamma(x) \setminus \{u,v\}$,
(iii) N_x has no induced claw centered at x,
(iv) G has no induced claw centered at t that contains the edge tx for any $t \in V(\Gamma(x) \setminus \{u,v\}$.

The $kl_{u,v}$ closure of G is obtained by creating a sequence of graphs $G = G_1, G_2, ..., G_n$ where G_{i+1} is obtained from G_i by completing the neighborhood of $x \in V(G \setminus \{u,v\})$ when N_x is a $kl_{u,v}$-cone. The maximal graph obtained by this process will be referred to as $kl_{u,v}(G)$.

This closure is especially useful because when applied to G it preserves the length of the longest u,v path and also preserves the property of G being P_ℓ-free. We summarize the important facts about this closure below when G is a claw-free graph:

Theorem 6. *[8] Let G be a claw-free graph and suppose $G' = kl_{u,v}(G)$. Then:*

(i) G' has a hamiltonian u,v path if and only if G has a hamiltonian u,v path,
(ii) If G is P_ℓ-free, so is G',
(iii) $G' \setminus \{u,v\}$ is the line graph of a triangle-free graph.

Proof. Fact (i) and the assertion that $G' \setminus \{u,v\}$ is a line graph are proven in [8]. We make the additional observation that when G is claw-free, then it is the line graph of a *triangle-free* graph. Kelmans showed that when G is a claw-free graph $G' \setminus \{u,v\} = c(G \setminus \{u,v\})$ where c is the standard closure of Ryjáček (see [9].) In his paper, Ryjáček noted that $c(G)$ is the line graph of a triangle free graph, which gives item (iii).

Item (ii) has already been noted in the closure of Ryjáček [9] and the proof here is identical.

We thus assume that $G = kl_{u,v}(G)$ for the remainder of the paper. Let $\tilde{G} = L^{-1}(G)$ denote the inverse line graph of $kl_{u,v}(G)$. A basic property of line graphs is the following:

Fact 1. Suppose G contains some graph H as a (not necessarily induced) subgraph. Then $L(G)$ contains $L(H)$ as an induced subgraph.

Based on this fact it follows that if a 3-connected, claw-free graph G is P_9-free, then $\tilde{G}$ does not have any copies of P_{10} as subgraphs.

Let $S = \Gamma(u)$ denote the neighborhood of u and $T = \Gamma(v)$. Let $R = L^{-1}(S)$ and $B = L^{-1}(T)$ be the set of edges in the inverse line graph $\tilde{G}$ corresponding to the set of vertices S and T in G respectively. We call the edges in R 'red' and edges in B 'blue'. Of course an edge may be both red and blue, i.e. $R \cap B$ is not necessarily empty. The key to our approach is the following:

Fact 2. Suppose there is a dominating trail in $\tilde{G}$ starting at a red edge $r \in R$, and ending at a blue edge $b \in B$ with $b \neq r$. Then there is a u, v hamiltonian path in G.

By a dominating trail starting at a red edge r and ending at a blue edge b, we mean a dominating trail starting at one end of r and ending at one end of b, which does not include either r or b in the trail itself. It is well known that a dominating trail in a graph corresponds to a hamiltonian path in the corresponding line graph. By starting adjacent to a red edge, and finishing adjacent to a blue edge we ensure that the endpoints of the hamiltonian path lie in the neighborhoods of u and v, and hense the path may be extended from a hamiltonian path in $L(\tilde{G})$ to a u, v hamiltonian path in G.

Our task throughout the remainder of the section is to show that any given inverse line graph $\tilde{G}$ with longest path length at most nine has a dominating trail from some $r \in R$ to some $b \in B$. Let $P' = p_0p_1p_2 \dots p_{k-1}$ denote a longest path in $\tilde{G}$, and let $P = p_1p_2 \dots p_{k-2}$ be the vertices in the interior of P'. Note that $|P'| < 10$, so $|P| \leq 7$.

We will choose to view $\tilde{G}$ so that P runs from left to right and, since $\tilde{G}$ is connected, the rest of the graph $\tilde{G} \setminus E(P)$ will be a disjoint set of components of $\tilde{G}$ attached to P' at a number of the p_i, $i \in \{0, 1, \dots, 9\}$. We will further classify these components by the number of vertices along P' on which the component is attached as follows:

Lemma 1. *Let $\mathcal{C}$ consist of the set of components of $\tilde{G} \setminus E(P)$. Then a component $C \in \mathcal{C}$ can be classified as one of four types T_1, T_2, T_3, and T_4, where $C \in T_i$ if $i = |C \cap P|$. Furthermore, if $C \in \mathcal{C}$ contains more than one vertex that is not in P the following are true:*

(*i*) *If C has exactly one edge incident with P, i.e. C is of type (a) or (b), then there is both a red and blue edge in C (they may be the same). Furthermore, such a component is unique if it exists.*
(*ii*) *If C has exactly two edges incident with P, then there is either a red or a blue edge within C. In this situation, C must be of type (e), (g), (h), or a special case of type (c), (d) or (f). There may be at most two components of such a type for each color.*
(*iii*) *If $C \in T_1$ and has more than one edge incident with P, i.e. C is of type (c), (d), or (e), then there exists a closed trail dominating C.*

Furthermore, all components C are isomorphic to one of the types enumerated in Appendix A.

Proof. That the components fall into one of the types $\{T_i\}_{i=1}^4$ is immediate from the fact that $|P| \leq 7$, and $\tilde{G}$ is triangle free. The first assertions of (*i*) and (*ii*) follow from G being 3-connected. Consider $e, e_1, e_2 \subseteq \tilde{G}$. Define $L(e)$ to be the vertex in G that corresponds to the edge e in $\tilde{G}$. Suppose the removal of e from $\tilde{G}$ disconnects e_1 and e_2. Since G is 3-connected, the removal of $L(e) \in V(G)$ does not disconnect $L(e_1)$ and $L(e_2)$. In fact, there must exist two vertex disjoint paths between them in G as $G \setminus L(e)$ is at least 2-connected. Since these paths do not correspond to paths in $\tilde{G}$, they must pass through u and v (and hence there must be red and blue edges on both sides of the cut). Likewise, if the removal of two edges disconnects $\tilde{G}$, the three connectivity of G implies that there must be either a red or a blue edge on each side of the cut. (Note: if $uv \in E(G)$, then it may be the case that there is a red edge on one side of the cut, and a blue edge on the other - there need not be the same color on both sides.)

The second part of (*ii*) follows from the fact that there cannot be three vertex disjoint red (or blue) edges. If there were, this would correspond to a claw in G. This observation also helps imply the second part of (*i*). Suppose there were two such components, connected to P by e_1 and e_2. There cannot be a red or blue edge elsewhere in the graph, other than in those components, as otherwise there would be three disjoint edges of the same color. But then removing $L(e_1)$ and $L(e_2)$ disconnects u and v from the remainder of the graph, violating the condition that G is 3-connected.

Statement (*iii*) follows from inspection of the types enumerated in Appendix A. We also argue that these are the only classes in the appendix.

A brief outline of the proof is as follows: in the case where there are T_3 or T_4 components, it turns out that the structure of $\tilde{G}$ is very restricted when we take into consideration that P is a longest path. We will (through some rather tedious, but easy, case analysis) handle these cases separately. In the case where all components are of type T_1 or T_2, we will define a new (multi)graph on a subset $S \subseteq P$, where a suitably defined trail hitting all vertices of S (and perhaps a few specified edges) will correspond to a dominating trail starting at a red edge and ending at a blue edge in $\tilde{G}$. By the observation in Fact 2, this dominating trail in $\tilde{G}$ will correspond to the desired hamiltonian uv-path in G. This will be the main subject of the next section.

2.2. Graphs without T_3 and T_4 components

Here, as in the previous section, we assume $\tilde{G}$ is a triangle-free graph with maximum length path P and $\mathcal{C}$ is the set of components of $\tilde{G} \setminus E(P)$. By assumption, all components in $\mathcal{C}$ are of type T_1 or T_2.

Let $S \subseteq V(P)(\subseteq V(\tilde{G}))$ denote the set of vertices $x \in V(P)$ satisfying either of the following:

(i) x is incident to an edge in some component $C \in \mathcal{C}$.
(ii) All edges incident to x are colored.

We define a (multi)graph on S where there is an edge $xy \in E(S)$ with multiplicity the total multiplicity of any of the following events:

(i) There is an edge from x to y in $\tilde{G}$.
(ii) There is a T_2 component C such that $V(C) \cap V(P) = \{x, y\}$.
(iii) If $x = p_i$ and $y = p_{i+2} \in V(P)$ and $p_{i+1} \notin S$.

Note that we call an edge $e \in S$ *marked* if it arises from case (*ii*) above and the component C contains an edge e' which is vertex disjoint from P. The reason that we mark these edges is because it will be necessary that our dominating trail in S uses these edges in order to lift to a trail in $\tilde{G}$ which dominates edges such as e'.

We color an edge $xy \in E(S)$ red (respectively blue) if it corresponds to a red (resp. blue) edge in P or if it corresponds to a component $C \in \mathcal{C}$ that contains a colored edge.

Note that by part (*ii*) of Lemma 1, all marked edges are colored. An edge, of course, may be colored both red and blue.

We will now use the colored edges to color the vertices of S. The following lemma is key in allowing us to do so in a well-defined manner:

Lemma 2. *If an edge in $\tilde{G}$ is colored, then one of its endpoints must be the center of a monochromatic star.*

Proof. Let xy be a colored edge. Without loss of generality, we can assume that it is colored red. If all edges incident to x are red, then x is the center of a monochromatic star. So assume that there exists an edge xx_i such that xx_i is not colored red. If there exists an edge yy_j such that yy_j is also not red, then the images of xy, xx_i, and yy_j in $G = L(\tilde{G})$ along with the vertex u form an induced claw centered at the image of xy. Therefore, all edges yy_j must be colored red. The argument for an edge colored blue is similar, with v being the fourth point in the induced claw.

We color a vertex $x \in S$ red (or blue) if one of the following conditions hold:

(i) x is the center of a monochromatic star in red (or blue),
(ii) x is adjacent to a marked red (blue) edge.
(iii) x is the attachment point of a component of type (a) or (b) in $\tilde{G}$.

Notice that in case (*iii*) above, x is colored both red and blue since, by Lemma 1 (i), the component must contain an edge of each color. Our main tool is the following:

Lemma 3. *If there exists a spanning trail T in S with all of the following properties:*

(i) One endpoint of T is a red vertex and the other is a blue vertex,
(ii) If a component of type (a) or (b) is present and $y \in V(S)$ is the attachment point of this unique component in $\tilde{G}$, then T starts at that vertex (which is colored both red and blue),
(iii) T uses every marked edge;

Then there exists a dominating trail in $\tilde{G}$ connecting a red edge to a blue edge.

Proof. There is an obvious lift from such a trail T to a trail T' in $\tilde{G}$. It is clear that T' starts at a red vertex because one of the three conditions listed above. The first possibility is that T starts at a red vertex that corresponds to the center of a red star in $\tilde{G}$ then the trail T' starts at that vertex. The second possibility is that T starts at a red edge that corresponds to a marked component C which contains a red edge. In this case we will consider T' as starting at the red edge within C. By Lemma 1 and inspection we observe that we can find trail starting at a red edge in the interior of any component of type T_1 or T_2 that dominates all edges of C with the exception of possibly one edge incident to P.

At a given vertex x in $\tilde{G}$, there may exist components of type T_1. Note that except for the case when x is colored both red and blue, there is a loop based at x which travels through the component and dominates all edges. In the case where there is a T_1 component connected to P by a single edge, we start at the red edge within this component in order to dominate it. Thus, we may extend T' to a trail T'' which dominates all T_1 components.

Note that all T_2 components which are not a single vertex connected to P by two edges are marked, and thus are used by assumption. An edge in S that is not used by T may correspond to an edge in $\tilde{G}$ or may correspond to a T_2 component consisting of a single vertex connected to P by two edges. In these cases the edges involved are dominated as their endpoints are in S and all vertices in S are visited. Another possibility is that an edge in S not used by T corresponds to a $P_3 \subseteq P$. Note that Lemma 1 guarantees that this edge cannot correspond to a longer path. The end vertices of this path are clearly visited by T', which shows that the edges of this P_3 are dominated by T'.

For the rest of the section, we concentrate on finding the desired trail T in S which uses all of the vertices of S and all of the marked edges.

We will begin by assuming that $V(S) = V(P)$. That is, none of the vertices on the main path are lost when we perform the reduction from $\tilde{G}$ to S. We will then remark on how the case when $|V(S)| < |V(P)|$ is different. Let $s_1, s_2, \ldots, s_7$ denote the vertices of S and define the graph S' so that $V(S') = V(S)$ and $E(S') = E(S) \setminus \{s_i s_{i+1} : i = 1, 2, \ldots, 6\}$. Note that since we have removed more than three edges from S then S' may be disconnected.

Our main argument is to find an appropriate connected subgraph in which all of the vertices are of even degree and there is an edge between a red and blue vertex. The even degree condition implies that this subgraph has an eulerian trail. Futher, since there is an edge between a red vertex and a blue vertex, we will thus find an eulerian trail from the red vertex to the blue vertex which finishes the argument. The main obstacle to this goal is that S' may be disconnected, and in fact might have up to five components. Since S is connected (in fact essentially three-edge connected) we can easily reintroduce enough edges to connect S' with the added edges. The argument is more complicated however due to the fact that we wish to preserve the parity of the degrees of all of the vertices as well. Therefore, we split the remainder of the argument into cases based on the number of components that have at least one color.

Definition 1. *Let G be a graph and $H \subseteq G$ be a subgraph such that $V(H) = V(G)$. Then if H has at least one vertex of odd degree, we* rectify *H by adding edges from $E(G) \setminus E(H)$ in order to make the degree of every vertex even. In this case we call H* rectifiable.

Notice that not every subgraph is necessarily rectifiable. However we will now show that S' is always rectifiable.

Lemma 4. *It is always possible to rectify S' by reintroducing a subset of the edge set $\{s_i s_{i+1} : 1 \leq i \leq 7\}$.*

Proof. It is a simple fact that the number of vertices in S' with odd degree must be even. We will therefore pair up the vertices of odd degree, (s_i, s_k), consecutively along the path from left to right and define a new graph S'' so that $V(S'') = V(S')$ and

$$E(S'') = E(S') \cup \{s_j s_{j+1} : (s_i, s_k) \text{ are a pair of vertices with odd degree and } i \leq j < k\}$$

Note that if s_j is of odd degree in S', then this process increases the degree sum by one, thereby making it even. Also note that if s_j is of even degree and $i < j < k$ then the degree of s_j is increased by two, preserving its even parity. Thus every vertex in S'' is of even degree.

Theorem 7. *Any graph S' constructed as described above can be modified by adding an edge between a red vertex and a blue vertex so that the graph obtained by rectifying the modified S' contains an eulerian circuit.*

Proof. The argument proceeds by cases depending on the number of components that contain a colored vertex, and we begin with a simple case that motivates the ideas used throughout the proof.

Case 1: (S' is connected)
We start by adding an edge between a red vertex and a blue vertex, which we will refer to as a *phantom edge* since the edge itself does not exist in $\tilde{G}$. The purpose of the phantom edge is to directly connect a red and blue vertex, so if they are already connected this edge is not strictly necessary. Finally since S' (and hence S'') is connected and every vertex in S'' has even degree by Lemma 4, it follows that we can find an eulerian circuit in S''. Note that since S'' has an edge between a red and a blue vertex, if we remove this edge from S'' the result is an eulerian trail starting at a red vertex and ending at a blue vertex. Thus we have found the desired trail when S' is connected.

For the rest of the proof we will assume that S' is disconnected, and split the argument into cases based on the number of components of S' which have at least one colored vertex. For each case, we argue that there is a way to connect S'' by strategically adding the phantom edge, removing non-marked edges, and rectifying S'.

Case 2: (Exactly one component of S' contains colored vertices)
In this case, S' can contain at most three components. Otherwise if there are four or more components, a simple averaging argument yields that at least one component must be an isolated vertex. Any isolated vertex in S' can be disconnected from S by at most a 2-cut and hence must be a colored vertex in order to retain the 3-edge connectivity of $\tilde{G}$. By the assumption of the case, this must be the only colored vertex in S. This implies that all of the colored vertices are on one side of a 2-edge cut in S. This yields a 2-cut in G which cannot happen. Thus there can be at most three components. Note that we add the phantom edge only to connect a red vertex to a blue vertex, but otherwise we do not use the phantom edge in this case to increase connectivity among the components of S'. Also note that since we can place the phantom edge between any red and blue vertex, if there is a vertex that is colored both red and blue due to a component of type (a) or (b) we take care to ensure that it is one of the endpoints of the phantom edge.

If there are two components, let us call them C and N where C is the component which contains the colored vertices. Since C and N cannot be separated by a two-edge cut, there must be a vertex from C between two vertices from N (where “between” here is in reference to the ordering given to the vertices of S' by the main path). Thus consider the subpath $s_i s_{i+1} \dots s_j$ such that $s_i, s_j \in V(N)$ and $s_k \in V(C)$ for $i < k < j$. If, after rectifying S', either of the edges $s_i s_{i+1}$ or $s_{j-1} s_j$ are added, then the two components will be connected and we can find the desired eulerian circuit. Otherwise, neither edge is added in the process of rectification. We can remedy this situation as follows. Since N is a connected component, there exists a path in N connecting s_i to s_j. We will remove from S' the edge of the shortest such path which passes over the edges $s_i s_{i+1}$ and $s_{j-1}s_j$ in our embedding. Note that there must be a single edge which passes over both since there are no vertices of N strictly between s_i and s_j. Since no edges in N are colored by assumption, then the edge which we remove is not marked, thus its deletion causes no problems. It can easily be observed that this changes the parity of exactly one vertex to the left of s_{i+1} by one as well as the parity of exactly one vertex to the right of s_{j-1}. Therefore, when we rectify S', the edges $s_i s_{i+1}$ and $s_{j-1}s_j$ will now be added, which connects N and C. Note that by removing an edge of N, we may disconnect it into a most two subcomponents, but if this happens s_i and s_j will be in different components and both are connected to C in the process. Thus the result of this removal prior to rectifying S' is a single component after rectifying S', and hence we can find the desired eulerian trail.

If there are three components, then the orders of the three components must be precisely two, two and three. This follows from the fact that none of the components are allowed to have order one without there being a 2-edge cut between the component with colored vertices and at least one of the components without colored vertices. The vertices $\{s_0, s_1, s_5, s_6\}$ must span at least two components since there are four vertices. Thus at least one of the edges (s_0, s_1) or (s_5, s_6) connects two components of S'. By forcing either s_0 or s_6, respectively, to have odd degree in S', we can force two components to be connected by rectifying the modified S'. Consider then the outer vertices along the main path, s_0 and s_6. If these two vertices are in the same component, then in order to avoid the 2-edge cut (s_0, s_1) and (s_5, s_6) the vertices must be in the unique component of order three. An easy observation is that we may make the degree of either s_0 or s_6 (of our choice) odd by removing edges without disconnecting the component, since no edge incident to either s_0 or s_6 is marked. Otherwise, if s_0 and s_6 are in different components, then at least one of the two vertices must be in a component of order two, let us say s_0 without loss of generality. Thus we can ensure that the degree of s_0 is odd. Note also that s_1 must be in a different component from s_0 since the only edge in S which connects the two vertices is the edge along the main path. (This follows from the fact that S is triangle-free.) In both cases the number of components is reduced from three to two. It is easy to observe that we may apply the argument from the previous paragraph to connect the third component. Note that the steps taken in the above argument cannot change the parity of the outer vertex which we modified.

Case 3: (Exactly two components of S' contain colored vertices)
Consider the two components of S' which have a colored vertex. Since there is at least one blue vertex and at least one red vertex in S', it follows that one of the components must contain a red vertex and the other a blue vertex. We therefore connect the two colored components by adding a phantom edge between the red vertex of one component and the blue vertex of the other component. Now there is exactly one component with colored vertices, so the resulting graph can be handled with Case 2. As in the previous case, we take care to choose any vertex colored both red and blue due to a component of type (a) or (b) as an endpoint of the phantom edge if such a vertex is present.

Case 4: (Exactly three components of S' contain colored vertices)
Let s_i, s_j, s_k, $0 \leq i < j < k \leq 6$ be the first colored vertex from the colored components C_1, C_2 and C_3 respectively as we traverse the main path of S from left to right. There must be some $\ell, i \leq \ell < j$ such that all of the vertices from s_i and s_ℓ are in C_1 and $s_{\ell+1} \in C_2 \cup C_3$. Similarly there must be some integer m, $i \leq m < j$ such that $s_m \in C_1 \cup C_3$ and $s_{m+1} \in C_2$.

First suppose that s_i and s_j are red and s_k is blue. In this case, if the edge $s_m s_{m+1}$ is not added when we rectify the graph, then by adding the phantom edge between s_i and s_k we force the edge $s_m s_{m+1}$ to be present after rectifying the modified S'. Hence all three colored components are connected. Suppose instead that the edge $s_m s_{m+1}$ is added by rectifying S'. Notice that, if $s_m \in C_1$ then simply choosing $s_j s_k$ as the phantom edge connects all three components, so suppose that $s_m \in C_3$. Since $s_k \in C_3$ then there is a path connecting these vertices. Consider the edge in this path which passes over the edge $s_m s_{m+1}$. This edge could not have been marked, or else s_m would have been colored, and we assumed that s_k with $k > m$ is the first colored vertex in C_3. Thus we remove this edge from S' and add the phantom edge again between s_i and s_k. The combined result of these two modifications to S' is that all three colored components are again connected after rectifying the modified S'. Finally we must consider the possibility of the presence of components without colored vertices. In order to satisfy the assumption that $s_m \in C_3$ for some $i < m < j$ we need that four vertices are in colored components. Since each noncolored component must contain at least two vertices and that there are only at most seven vertices total in S', it follows that there can be only a single non-colored component in this case, call this component N. The spanning subtree of N must be a path of length two, so remove all of the "excess" edges, so that N is isomorphic to a path of length two. Then two of the vertices in N are both of odd degree, which leaves only two possibilities. Either the two vertices in N of odd degree are consecutive along the main path of S, or else they are not. Each of the two vertices will be connected to the vertex to either its immediate left or immediate right after rectifying S', so in the latter case, N will always be connected to one of C_1, C_2 or C_3. In the former case, it is possible that the two vertices in N of odd degree are paired up in the rectification process. While this initially seems problematic, we can fix this by removing one of the two edges and isolating the corresponding vertex. The two vertices will each be connected to a colored component. Thus we conclude that we can connect all components after rectifying S' in this case.

Now suppose instead that s_i and s_k are red and s_j is blue. If there exists a pair of vertices $s_\ell, s_{\ell+1}$ such that $s_\ell \in C_1$ and $s_{\ell+1} \in C_3$ (or vice versa), then whether or not the edge $s_\ell s_{\ell+1}$ is added after rectifying S', we can add the phantom edge in such a way as to connect all three colored components. If no such pair exists, then there must be an integer m such that $j < m < k$ and $s_m \in C_2$, $s_{m+1} \in C_3$. Since s_{m+1} is not 1-edge disconnectable from S, there must be either a vertex from C_3 to the left of s_m or a vertex from $C_1 \cup C_2$ to the right of s_{m+1}. In either case, there is again a (necessarily unmarked) edge which passes over $s_m s_{m+1}$. Removing this edge allows us to force the edge $s_m s_{m+1}$ to be added after rectifying S' and thus connecting all of the colored components as before. Finally we can handle the noncolored components similarly to the previous paragraph.

There is one final case to consider, when one of the colored vertices is colored both red and blue. In this case the doubly colored vertex must be one of the vertices in the phantom edge. If s_i (or symmetrically s_k) is the vertex colored both red and blue, then s_i is a component of order one. There are two possible choices for the phantom edge, $s_i s_j$

and s_is_k, and the choice is dictated by the presence of the edge s_ms_{m+1} where m is defined as above. Instead, suppose that s_j is the component of order one which is colored both red and blue. It is easy to see in this case there must be an ℓ, $1 \leq \ell \leq 6$ such that $s_\ell \in C_3$ and $s_{\ell+1} \in C_1$ (or vice versa) from the fact that s_j is the only vertex in C_2 and neither C_1 nor C_3 can be 1-edge disconnected from S. By the symmetry present in this case, we may assume that $\ell < j$. If $s_\ell s_{\ell+1}$ is present after rectification, we choose s_js_k as the phantom edge. If $s_\ell s_{\ell+1}$ is not present, then we choose s_is_j as the phantom edge.

Case 5: (Exactly four components of S' contain colored vertices)
There are at most seven vertices in the graph S', and since there are four components with color, there must be a colored component which is an isolated vertex. Further this isolated component cannot be either of the outer two vertices (s_0 or s_6). Let us call s_i the isolated colored component. Since s_i is of even degree in S', it will not interfere with whether or not the edges $s_{i-1}s_i$ and s_is_{i+1} are included after rectifying S. Therefore, we can ignore this vertex and use the argument from Case 4 for the remaining graph $S' \setminus \{s_i\}$ which has precisely three colored components.

Thus there is always a way to modify the graph S' so that rectifying the modified S' results in a connected graph with a eulerian trail from a red vertex to a blue vertex.

2.3. Graphs with T_3 and T_4 components

Here, we again assume $\tilde{G}$ is a triangle-free graph with P the interior of a selected longest path P' in $\tilde{G}$. For ease of notation, we will use $\{0, 1, 2, \ldots\}$ to label the vertices of P'. Also, for any vertex x on P', we will use x^+ to denote the vertex immediately following x and x^- to denote the vertex immediately preceding x along P. If there is a choice of P' without a T_3 or T_4 component we may select that P' and be guaranteed a red-blue dominating trail from the previous subsection. We can therefore assume that for all choices of P' there is at least one T_3 or T_4. When referring to a specific component, we will refer to it as $T(\overline{X})$ where $\overline{X}$ is the set of vertices of P which the component is incident to. If there exists a vertex within the component that is not incident to P, we will denote the component by $T'(\overline{X})$.

In each subsection, we will consider one component to be the *primary component* and all other components *secondary components*. Throughout, we will use A when referring to a vertex in the primary component that is adjacent to P and B when referring to a vertex in the primary component that is not adjacent to P. We denote by $i\overrightarrow{P}j$ the subpath of P between vertices i and j and denote by $i \ldots j(k \ldots \ell)$ either the trail $i \ldots j$ or the extended trail $i \ldots jk \ldots \ell$.

The claw-free condition on G forces a useful structure on the colored edges of $\tilde{G}$. We will use this structure to color the vertices of $\tilde{G}$ and more easily identify red-blue dominating trails.

We initially color vertices of P based on the colored edges in $\tilde{G}$. A vertex is colored red or blue if it is the center of a monochromatic star of that color. If an edge is colored, at least one of its end vertices is also colored by Lemma 2. If a dominating trail that starts at a red vertex and ends at a blue vertex can be found, it can be converted to a trail that begins at one end of a red edge and ends at one end of a blue edge that does not use either edge in the following way. If there is an edge incident to the colored vertex that is not used in the dominating trail, we have the necessary trail. If there is not an edge incident to the colored vertex that is not used in the dominating trail we truncate the path at the second to last vertex to ensure we end at the end of the colored edge without actually using it. Note that truncating the trail in this manner only removes dominance of edges incident to an end vertex, all of which are used in the trail if this truncation is necessary.

While only one type of T_4 component, $T(1,3,5,7)$, can exist, eleven types of T_3 components can exist: $T(1,3,5)$, $T(1,3,6)$, $T(1,3,7)$, $T(1,4,6)$, $T(1,4,7)$, $T(1,5,7)$, $T(2,4,6)$, $T(2,4,7)$, $T(2,5,7)$, and $T(3,5,7)$. Additionally, different choices of P' can give rise to different component types. Therefore, we choose P' in a specific manner so as to obtain one of our preferred primary components if possible.

We select our P' by carefully reducing the set of longest paths. We select first the subset of longest paths that have the maximum number of T_4 components. We further reduce the number of paths in our set by taking the subset with the fewest T_3 components. From this final subset we choose a path that has the most preferred primary component. Our preference for a primary component is given by the following order: $T(1,3,5,7)$, $T(1,3,7)$, $T(1,3,5)$ and finally $T(2,4,6)$. We need not extend our component preferences further because if any of the other seven T_3 types is present, we can find an alternate P' that has one of our preferred types and the same number of T_3 components.

We can eliminate the need to consider the components $T(3,5,7)$, $T(2,5,7)$, $T(2,4,7)$, and $T(1,5,7)$ by noting that reversing the path P' transforms them to $T(1,3,5)$, $T(1,3,6)$, $T(1,4,7)$, and $T(1,3,7)$, respectively. We can also trade the components $T(1,3,6)$, $T(1,4,6)$ and $T(1,4,7)$ in favor of $T(2,4,6)$, but this requires a bit more work. In each of the following cases, we first consider an alternate path with the intention of forcing certain edges to be present. We then show there exists an alternate path with the preferred component.

Suppose $T(1,3,6)$ is present and the number of T_3 components is minimized. The alternate path $01A34567(8)$ must have a T_3 with center 2 because the T_3 with center A was eliminated. Vertex 2 is adjacent to 1 and 3 and must be adjacent to a third vertex. Edges $\overline{02}$ and $\overline{24}$ create triangles, and edges $\overline{25}$, $\overline{27}$, and $\overline{28}$ create longer paths $012543A7(8)$, $01A654327(8)$, and $01A6543287$ respectively. The final edge must then be $\overline{26}$. The alternate path $543A12678$ must have a T_3 with center 0 because the T_3 with center A was eliminated. Edge $\overline{02}$ creates a triangle. Edges $\overline{04}$, $\overline{05}$, $\overline{07}$, and $\overline{08}$ create longer paths $A32104567(8)$, $054321A67(8)$, $(8)70123456A$, and $780123456A$ respectively. The remaining edges are then $\overline{03}$ and $\overline{06}$, and the T_3 with center 0 is of type $T(2,4,6)$ relative to this alternate path.

Suppose $T(1,4,6)$ is present and the number of T_3 components is minimized. The alternate path $01234A67(8)$ must have a T_3 with center 5 because the T_3 at A was eliminated. Vertex 5 is adjacent to 4 and 6 and must be adjacent to a third vertex. Edges $\overline{35}$ and $\overline{57}$ create triangles, and edges $\overline{05}$, $\overline{25}$, and $\overline{58}$ create longer paths $054321A67(8)$, $01A432567(8)$, and $01234A6587$ respectively. The final edge must then be $\overline{15}$. The alternate path $23451A678$ must have a T_3 with center 0 because the T_3 with center A was eliminated. Edge $\overline{02}$ would create a triangle. Edges $\overline{03}$, $\overline{05}$, $\overline{07}$, and $\overline{08}$ create longer paths $0321A4567(8)$, $054321A67(8)$, $(8)70123456A$, and $780123456A$ respectively. The remaining edges are then $\overline{04}$ and $\overline{06}$, and the T_3 with center 0 is of type $T(2,4,6)$ relative to this path.

Suppose $T(1,4,7)$ is present and the number of T_3 components is minimized. The alternate path $654321A78$ must have a T_3 with center 0 because the T_3 at A was eliminated. Vertex 0 is adjacent to 1. Edge $\overline{02}$ would create a triangle. Edges $\overline{03}$, $\overline{05}$, $\overline{06}$, and $\overline{08}$, create longer paths $0321A45678$, $234A105678$, $0654321A78$, and $087654321A$ respectively. The remaining edges are then $\overline{04}$, and $\overline{07}$. Repeating this argument with the original path reversed implies that edges $\overline{18}$ and $\overline{48}$ are also present. The original $T(1,4,7)$ based at A then becomes type $T(2,4,6)$ relative to alternate path 321048765.

Note that in all three cases, any secondary T_3 meets the path at three vertices in the set $\{1,2,3,4,5,6,7\}$, which also appear in the alternate paths and that no new secondary T_3 can be created because no new vertices are added to form the center of a secondary T_3. The alternate paths must then also belong to the subset of longest paths with the fewest T_3 components and so there is indeed a path with one of our preferred components.

We now turn our attention to finding red-blue dominating trails in $\tilde{G}$ for each preferred primary component. If we can find a dominating trail between any two potentially colored vertices from the set $\{1,2,3,4,5,6,7,A\}$, then no matter which vertices are in fact colored we can find a red-blue dominating trail.

2.3.1. Graphs with primary component $T(1,3,5,7)$ We first show that no vertex can be distance two from P. The vertices $\{1,2,3,4,5,6,7,A\}$ form an 8-cycle. Clearly, any vertex of distance two from this cycle will form a P_{10}. Also note that the edge $\overline{08}$ creates a P_{10}, $80\overrightarrow{P}7A$, so all edges must be dominated by the vertices of P.

Since all edges are dominated by P, if a colored vertex x is not on P and it is necessary to start or end the dominating trail at that vertex, it is sufficient to find a dominating trail that starts or ends at a neighbor of x and extend the trail appropriately.

If the start and end vertex are the same or appear consecutively on P, $x\overrightarrow{P}7A1\overrightarrow{P}x^-(x)$ and $x\overleftarrow{P}1A7\overleftarrow{P}x^+(x)$ are dominating trails. For any remaining pair of red and blue vertices in $\{1,2,3,4,5,6,7,A\}$ one of the following provides a dominating trail: $(A)1\overrightarrow{P}7(A)$, $123A7\overleftarrow{4}(3)$, $1\overrightarrow{P}5A76(5)$, $21A7\overleftarrow{P}4$, $21A345A76(5)$, $21A3\overrightarrow{P}7(A)$, $321A345A76(5)$, $321A3\overrightarrow{P}7(A)$, $45A123A76$, $4\overleftarrow{P}1A567(A)$, $5\overleftarrow{P}1A567(A)$, and $67A5\overleftarrow{P}1A$.

2.3.2. Graphs with primary component $T(1,3,7)$ None of the reasoning in the $T(1,3,5,7)$ section made use of the edge $A5$ that $T(1,3,7)$ lacks. By the same argument used with a $T(1,3,5,7)$ primary component, all edges must be dominated by the vertices of P. The existence of a vertex x such that x is adjacent to 2,4, or 6, gives the longer path $x21A3\overrightarrow{P}8$, $0\overrightarrow{P}3A7\overleftarrow{P}4x$, or $x6\overleftarrow{P}1A78$ respectively. The edges 04, 60, 26, 28, and 48 give the longer paths $6540123A78$, $4560123A78$, $0126543A78$, $01A7\overleftarrow{P}28$, $0123A78456$, respectively. All other edges within the set $\{0,2,4,6,8\}$ create a triangle, thus all edges from $\{0,2,4,6,8\}$ must be to some subset of $\{1,3,5,7\}$. Thus the set $\{1,3,5,7\}$ dominates all edges in $\tilde{G}$, and a dominating trail need only contain these vertices.

As in the previous case, if a colored vertex x is not on P and it is necessary to start or end the dominating trail at that vertex, it is sufficient to find a dominating trail that starts or ends at a neighbor of x and extend the trail appropriately.

If the start and end vertex are the same or appear consecutively on P the following are dominating trails: $x\overrightarrow{P}7A1\overrightarrow{P}x^-(x)$ and $x\overleftarrow{P}1A7\overleftarrow{P}x^+(x)$.

The following provide dominating trails for several of the remaining possible pairs of end vertices : $(A)1\overrightarrow{P}7(A)$, $123A7\overleftarrow{4}(3)$, $1\overrightarrow{P}5A76(5)$, $21A7\overleftarrow{P}4$, $21A3\overrightarrow{P}7(A)$, $321A3\overrightarrow{P}7(A)$, $4\overrightarrow{P}A123A$, $5\overleftarrow{P}1A7$, and $567A123A$.

The only pairs not addressed above are $\{(2,5),(2,6),(3,5),(3,6),(4,6),(4,7),(6,A)\}$. These require us to examine a bit more of the structure of $\tilde{G}$. Since the number of T_3 components is minimized, the alternate path $A12345678$ must have a T_3 centered at 0. Edges $\overline{02}$, $\overline{04}$, $\overline{06}$, and $\overline{08}$ were already ruled out, so two of the remaining possible edges, $\overline{03}$, $\overline{05}$, and $\overline{07}$, must be present. If $\overline{05}$ is present, then $(2)3\overrightarrow{P}7A105$, $(2)3\overrightarrow{P}501A76$, $450123A76$, 43210567, and $67A10543A$ are dominating trails for the remaining pairs. If not, then $\overline{03}$ and $\overline{07}$ are both present and $(3)2107A345(6)$, $4\overrightarrow{P}7A32107$, and $6\overleftarrow{P}107A$ are dominating trails for the remaining pairs except $(4,6)$.

Lastly, consider the case of finding a dominating trail between 4 and 6 when $\overline{05}$ is not present. For such a dominating trail to be necessary, 4 and 6 must be the only red and blue vertices. Also, neither vertex is colored both red and blue, otherwise the trail that starts and ends at the same vertex will suffice. Without loss of generality, assume 4 is colored red. Vertex 4 is then incident to every red edge because Lemma 2 and our rules for vertex coloring imply that every red edge is incident to a red vertex and 4 is the only such vertex. By 3- connectivity of G, u has at least 3 neighbors and so there are at least 3 red edges. The only known edges incident to 4 are $\overline{34}$ and $\overline{45}$. By the previous analysis of edges from 4, the unknown third edge must be either $\overline{14}$ or $\overline{47}$ which produce dominating trails for $(4,6)$ of $4321A701456$ and $4321A7456$ respectively.

2.3.3. Modified Coloring for $T(1,3,5)$ and $T(2,4,6)$ Primary Components In the cases where the primary component is either a $T(1,3,5)$ or $T(2,4,6)$ we modify the coloring process to facilitate finding the required dominating trail. If a colored vertex y appears in a secondary component $T(\overline{X})$, we move the color to P by coloring every vertex in $\overline{X}$ with the color that appears on y. Similarly, if the first or last vertex of P' is colored, we move that color to the adjacent vertex on the interior of the path.

If we wish to use one of these newly colored x's as an end vertex of our dominating trail, we temporarily remove the component containing y before finding a dominating trail. Once we have the dominating trail in this new graph, we extend it to a dominating trail in the original graph by using a path within the removed component. The only components that can occur with either a $T(1,3,5)$ or a $T(2,4,6)$ where the path extension to the colored vertex may not necessarily dominate all edges in the component are e' and f'. These special cases are discussed at the end of this section.

In the event that we wish to use two newly colored vertices that received their colors from two different components as the end vertices of the dominating trail, we temporarily remove both components containing the colored y's. Once again, we can take the dominating trail in the new graph and extend it to a dominating trail in the original graph by using paths within the two removed components.

Lastly, it is necessary to address the case where it is necessary to use two of the newly colored vertices that received their color from the same secondary component. This is only necessary when all the colored vertices lie in a single component. The three connectivity of G requires there be three disjoint paths from the neighbors of u and v to the remainder of the graph, which forces the component to be either a T_3 or a T_4. The case where it is a T_4 was handled in a previous section, so we may assume it is a T_3. For the dominating trail, we choose distinct start and end vertices from the three possibilities. It is then easy to extend the trail to start and end at the appropriate place based on which edges are actually colored.

We now consider how to dominate the edges of e' and f' appropriately. We first note that the component e' has very specific color requirements. Since any two edges within the cycle of the component form a 2-cut, each set of pendant edges must be colored. Since there are at most two vertex disjoint edges of a single color due to the claw-free property of the original graph, two sets of pendant edges must be the same color while the third is different. Thus all edges of a single color are within this component and one of the colored vertices must be of distance one from P. It is then required that any dominating trail start at this component and end at the colored vertex outside of the component. Since a side vertex is colored with the correct color, the component can be dominated by going around the cycle and ending at that vertex.

The component f' cannot occur with the $T(1,3,5)$, and it can only occur with $T(2,4,6)$ when f' is adjacent to 2 and 6 and pendant edges within the component are not allowed. Every dominating trail that will be presented in the $T(2,4,6)$ section includes both 2 and 6, so the edge that is not dominated by the path extension is dominated by the original path.

2.3.4. Graphs with primary component $T(1,3,5)$ Unlike the previous two preferred components, the longest path need not be a P_9. While much of the following argument implicitly assumes a P_9 for P', these portions remain vacuously true if vertices 7 or 8 are not present.

First we consider what components can coexist with a $T(1,3,5)$. No vertex x can be adjacent to vertices 2 or 4 since $x21A3\overrightarrow{P}6(7(8))$ and $x4321A56(7(8))$ are longer paths. We also note that there cannot be a path of length 2 incident to vertex 3 since $xy321A56(7(8))$ would be a longer path. Lastly, if a vertex x is adjacent to both vertices 3 and 6 then $01A543x6(7(8))$ is a longer path and if a vertex is adjacent to both vertices 1 and 6 then $2345A1x6(7(8))$ is a longer path. This means that any T_2 or T_3 component must have all adjacencies from the set $\{1,3,5,7\}$, and all edges within these components must be dominated by the path. Further, it is possible to have T_1 components of type b and c incident with 5. The T_1 component e can also occur incident with 5, but the component cannot have any pendant edges on the vertices distance one from P. The T_1 component a can occur incident with either 5 or 6, and pendant edges can be incident to any vertex from the set $\{1,3,5,6,7\}$.

As stated above, a $T(1,3,5)$ can in theory coexist with $T(1,3,7)$, $T(1,5,7)$, and $T(3,5,7)$ components. However, the presence of any of these with a primary $T(1,3,5)$ imply the existence of a longest path with a $T(1,3,7)$. With the $T(1,3,7)$, P' itself is such a path. With the $T(1,5,7)$ the reversal of P' is such a path. If a $T(3,5,7)$ with center C is present, then relative to the alternate path $0123C7654$ the primary component, $T(1,3,5)$, is a $T(1,3,7)$ and this path would have been chosen instead of P'. This together with the arguments presented in the preceding paragraph implies that graphs with primary component $T(1,3,5)$ can only have $T(1,3,5)$ components as secondary T_3 components.

Now we consider what other structure a graph with a primary $T(1,3,5)$ component must have. The edges 04, 06, 08, 26, 28, and 48 give the longer paths $04321A5\overrightarrow{P}$, $A5\overleftarrow{P}106(7(8))$, $A5\overleftarrow{P}10876$, $01A5\overleftarrow{P}26(7(8))$, $01A5\overleftarrow{P}2876$, or $0123A54876$, respectively. Any other edge from within the set $\{0,2,4,6,8\}$ creates a triangle. Also, any edge from $\{0,2,4,A\}$ to vertex 7 would create a T_4 so we may assume that these edges are not present. Edges between pairs of vertices in the set $\{1,3,5\}$ together with the edges of the primary component form forbidden triangles, so $\overline{13}$, $\overline{15}$, and $\overline{35}$ cannot be present. The alternate paths $A1\overrightarrow{P'}$ and $01A3\overrightarrow{P'}$ must have T_3 components centered at 0 and 2 respectively. The restrictions already imposed give that both must be $T(1,3,5)$ components, and so the edges $\overline{03}$, $\overline{05}$, and $\overline{25}$ are present. These edges permit the alternate path $A32105\overrightarrow{P'}$ which must have a T_3 based at 4. The only option which does not produce a $T(1,3,7)$ is to include the edge $\overline{14}$, which creates another $T(1,3,5)$. The subgraph induced on $\{0,1,2,3,4,5,A\}$ is then a $K_{4,3}$ with partite sets $\{0,2,4,A\}$ and $\{1,3,5\}$. To make use of the symmetry of this $K_{4,3}$ we relabel the partite sets $\{a,b,c,d\}$ and $\{E,F,G\}$. For any choice of start and end vertices, a trail that dominates the entire $K_{4,3}$ structure is one of $aEbFcGdF(a)$, $(E)aGbFcGdE(b)$, and $EaGbFcGdF$.

It is now necessary to account for the end of the path P', as well as any additional structures that may be adjacent to vertex 5 or 6. If P' is a P_7, the only additional edges must be pendant edges incident with vertex 5. In this case all edges are dominated by the vertices of the $K_{4,3}$.

If P' is a P_8, any structure based at vertex 5 cannot contain a path of length three, otherwise there is a longer path. Any T_2 or T_3 secondary component based at 5 cannot also be adjacent to vertex 7 because it gives a longer path. Thus all T_2 and T_3 components must have their adjacencies within $\{1,3,5\}$ and all edges within the components are dominated by the vertices of the $K_{4,3}$. The only T_1 component that does not contain a path of length 3 is type (a). When considering the first 6 vertices of P' as a path, Lemma 1 gives that there can be at most one component of this type (a). So without loss of generality, we can assume that this component is the end of P' if it is present. If 6 or 7 is colored, the necessary dominating trail can be found by extending the dominating trail starting at a colored vertex within the $K_{4,3}$ and ending at 5 to include 6 and, if necessary, 7.

If neither 6 nor 7 is colored, 7 must have an adjacency within the $K_{4,3}$. There are two symmetric possibilities, either the adjacency is from the set $\{1,3\}$ or from the set $\{0,2,4,A\}$. Assume 7 is adjacent to 1 or 3. Let $\{E,F\}$ represent $\{1,3\}$, with F being the vertex 7 is adjacent to. The following trails give dominating trails for any pair of starting and ending vertices $aEbF765cFd(E(a))$, $aEbF765cFd(5)$, $EaF765bEcFd(E)$, $EaF765bEcFd(5)$, or $EaF765bFcEdF$. Now assume 7 is adjacent to a from the set

$\{0, 2, 4, A\}$. One of the following dominating trails is the desired trail: $a765bEcFdE(a(5))$, $a765bEcFdE(b, (F))$, $Ea765bFcEdF$, $Ea765bFcEd5$, $Ea765bEcFdE$, or $567aEbFcEd5$.

Now consider when P' is a P_9. If there is are components of type (c) or (e) adjacent to 5, we simply add the trail beginning and ending at 5 that dominates the component into the middle of the trail that dominates the remainder of the graph. Lemma 1 gives that there is at most one component of type (a) or (b). If there is a component of type (b) incident to 5 or a component of type (a) incident to 6, we take this component to be the end of the path. If 7 or 8 is colored, the necessary dominating trail can be found by extending the dominating trail starting at a colored vertex within the $K_{4,3}$ and ending at 5 to include 6 and 7, and, if necessary, 8.

If neither 7 nor 8 is colored, 8 must have an adjacency in the $K_{4,3}$. The adjacency cannot be from the set $\{0, 2, 4, A\}$ since that would result in the longer path $a8765b1c3d$. If 8 is adjacent to 5, we can dominate the end of the path by replacing one instance of the vertex 5 in the dominating trail with 56785. Lastly, if 8 is adjacent to 3, the alternate P_9 given by 012387654 transforms the primary component to a $T(1, 3, 7)$.

2.3.5. Graphs with primary component $T(2, 4, 6)$ Like the previous primary component, a $T(2, 4, 6)$ need not have a P_9 as the longest path. However, if the longest path is a P_8 the component can be viewed as a $T(1, 3, 5)$ when looking at the reversed path.

We first consider what additional structure must be present with this primary component. Edges between pairs of vertices in the set $\{1, 3, 5, 7\}$, with the exception of $\overline{17}$, form forbidden triangles if they differ by two or create one of the following longer paths: $0\mathbf{15}432A678$ or $012A654\mathbf{37}8$. The alternate paths $012A45678$ and $01234A678$ must have a T_3 with center 3 and 5 respectively. The only available edges to form these T_3 components are $\overline{36}$ and $\overline{25}$ respectively. The induced subgraph on $\{2, 3, 4, 5, 6, A\}$ is then a $K_{3,3}$ with partite sets $\{2, 4, 6\}$ and $\{3, 5, A\}$. Up to relabeling, the dominating trails for this $K_{3,3}$ are $aDbEcF(a)$ or $aDbEcFb$. Note that this $K_{3,3}$ structure forbids $\overline{24}$, $\overline{26}$, and $\overline{46}$.

Earlier we noted that $\overline{17}$ can be present, however this case is highly restrictive. If there is an edge from $\{2, 3, 4, 5, 6, A\}$ to a vertex, x, not in $P' \cup \{A\}$, then $\overline{17}$ cannot be present. Suppose that such an edge and $\overline{17}$ were both present. We either produce a path longer than P': $x32A456710$, $x5432A6710$, $xA23456710$, or have an alternate longest path with a $T(1, 3, 5)$ component: $x23456710$, $x43256710$, $x65432178$ both contradictions. The only additional vertices that can be present are pendent vertices from 1 or 7. Therefore edges incident to $\{3, 5, A\}$ are dominated by $\{2, 4, 6\}$. In this case, $1\overrightarrow{P}7$ suffices as a dominating trail provided both 1 and 7 are colored. If one of the 1 and 7 is colored and the other is not, without loss of generality we may assume 7 is colored and 1 is not, then we extend the dominating trail from a vertex v of the opposite color within the $K_{3,3}$ to 2 by beginning with the dominating trail within the $K_{3,3}$, that begins at v and ends at 2 and appending 217 to the end. Now consider when neither 1 nor 7 is colored. Let $\{D, E, F\}$ represent the partite set $\{3, 5, A\}$. The following dominating trails then suffice for any choice of starting and ending vertices: $D2176E4(F(2))$, $D2176E4(F(6))$, $2176D2E4$, $6712D6E4$, $2176D4E6$.

We may now assume that $\overline{17}$ is not present for the rest of the analysis on this primary component. We are much less restricted in additional components, but can still make several observations. If there exists a P_3 in a component with an endpoint from the set $\{3, 5, A\}$ or a P_4 in a component with an endpoint from the set $\{2, 4, 6\}$ we produce a longer path, e.g. $xy32A45678$ or $xyz456A210$ respectively. This observation gives that the only possible components incident to $\{3, 5, A\}$ are pendent edges, and the only components incident to $\{2, 4, 6\}$ are T_1 components of type (a), T_2 components of type (f) (where the type (f) component has no pendant edges) or type (g), or T_3 components of type (i). Note that if there is a component of type (a) incident to 2 or 6, we can assume that it serves as the end of the path P.

For the first part of the analysis, we assume that there are no secondary T_1, T_2 and T_3 components. We will account for these components at the end of this section. Let $\{D, E, F\}$ denote the partite set $\{3, 5, A\}$. If we wish for the path to begin at 1 and end at 7, we use the path $12D4E2F67$. If one is colored and either 7 is not colored or we wish to end at a vertex within the $K_{3,3}$ because it is incident to a type (a) component, it must be the case that 7 has an adjacency other than 6 within the $K_{3,3}$. To prevent a triangle, this adjacency must be 2 or 4. If 7 is adjacent to 2, we can use one of the following dominating trails: $12D672E4F(6)$, $12D672E6F(4)$, or $12D6E4F672$. If 7 is adjacent to 4, the following dominating trails cover all possibilities: $12D476E4F(2/6)$ and $12D476E2F(4)$.

The only case in which we would need to have both the starting and ending vertices within the $K_{3,3}$ would be if neither 1 nor 7 is colored. In this case, both 1 and 7 must have a second adjacency from within the set $\{2, 4, 6\}$. Both 1 and 7 and their incident vertices act the same as T_2 components, so we set them aside to be accounted for with the other T_2 components and use the appropriate path to dominate the $K_{3,3}$.

We now account for the secondary components previously set aside. By Lemma 1, only one type (a) component can be present, and if present it must have a red and a blue vertex. There must also be a red and a blue vertex not in the type (a) component. In this case we extend the dominating trail of the $K_{3,3}$ ending at the vertex incident to the component to a trail ending at the appropriately colored vertex in the type (a) component.

Recall that it was previously argued that any T_2 or T_3 component must be incident to vertices within the set $\{2, 4, 6\}$. Let $x, y \in \{2, 4, 6\}$. For a component $T(x, y)$, let Q_i denote a trail that travels through that component. If there are an even number of components $T(x, y)$, we can dominate these components by adding xQ_1yQ_2x to the middle of the dominating trail the first time we see x. An even number of T_3 components can be dominated in a similar manner. This means there are at most 1 of each of the following types of components that cannot be dominated in this manner: $T(2, 4)$, $T(2, 6)$, $T(4, 6)$ and $T(2, 4, 6)$.

If there is a component of type $T(2, 4, 6)$ and any T_2 component $T(x, y)$, then these can be paired and dominated in the above fashion. So if there is an unpaired $T(2, 4, 6)$ component, it must be the only component that cannot be paired and it can be treated as a T_2 since it has a T_2 subgraph. If all three of the possible T_2 components remain, they can be dominated by adding $2Q_14Q_26Q_32$ into the dominating trail. Lastly, if two T_2 components remain they can be treated as one T_2 component. For example, a $T(2, 4)$ and a $T(4, 6)$ will act the same as a $T(2, 6)$ component since the path $2Q_14Q_26$ can be added to the middle of the dominating trail in the same way $2Q_36$ can. Therefore, it is only necessary to consider modifying the dominating trails to account for one additional T_2 component.

When the dominating trail started at 1 and ended at 7, the new dominating trails are $12Q4D2E4F67$ for the component $T(2,4)$ and $12Q6D2E4F67$ for the component $T(2,6)$. The case where the component is $T(4,6)$ is symmetric to $T(2,4)$.

Now consider when the dominating trail began at 1 and ended within the $K_{3,3}$. Recall that in this case 7 had an additional adjacency to either 2 or 4. The additional T_2 must have at least one adjacency in common with 7. If the additional T_2 has the both adjacencies in common with 7, we add $67xQ6$, where $x \in \{2,4\}$, into the middle of $12DaEbF(c)$. If the T_2 has only one adjacency in common, we treat the edges incident to 7 and the T_2 as one component in the same manner as discussed previously. If this combined component is $T(2,6)$, we can use one of the following dominating trails: $12D672E4F(6)$, $12D672E6F(4)$, or $12D6E4F672$. If the component is $T(4,6)$, the following dominating trails cover all possibilities: $12D476E4F(2/6)$ and $12D476E2F(4)$. Lastly, if the component is $T(2,4)$ we can use one of $124D6E2F(4,6)$ or $124D6E4F(2)$.

In the case where the dominating trail starts and ends within the $K_{3,3}$, we modify the dominating trails as follows. If the component is $T(a,b)$ and the third vertex from the partite set $\{2,4,6\}$ is c, the following dominating trails give all possibilities: $DaQbEcF(a)$, $DaQbEaF(c)$, $aDbQaEcF(a,b)$, $aDbQaEbF6$, and $cDaQbEaFc$.

3. Appendix

Let G be a 3-connnected $(K_{1,3},P_9)$-free graph. In this section we want to classify all possible types of components that can be obtained from the inverse line graph of the closure $kl_{u,v}(G)$, namely $\tilde{G}$. As shown in Lemma 1, there are four types of components, T_1, T_2, T_3 and T_4.

For the components that lie in T_n, with $n \leq 4$, we say that the sequence $(x_0, x_1, \cdots, x_n)$ is admissible for component $C \in T_n$ if $\sum_{i=0}^{n} x_i = 8$ and component C can be attached to P with distance x_i between the $i-1$st and ith attachment point along the main path P. Here x_0 is the length between the left endpoint of the path and the first attachment point, and similarly x_n is the distance between the last attachment point and the right endpoint of the path. Some general observations about admissible sequences are:

1. $x_0, x_1 \geq 1$, and $x_i \geq 2$ for $1 < i < n$.
2. If ℓ is the maximum distance between two vertices in a component which are attached at attachment points i and j then $\displaystyle\sum_{k=i}^{j} x_k \leq \ell$.
3. If $x_0 = 1$ or $x_n = 1$ then the component type is trivial. That is, the component is a single vertex attached to the path P. In general for component C, it must hold that for any vertex $v \in C$, the distance between the left attachment point and v is at most x_0 and the distance between the right attachment point and v is at most x_n..

4. If component C is attached at vertices i and j then the maximum distance between i and j through the component must be at most equal to the distance between i and j along the path.

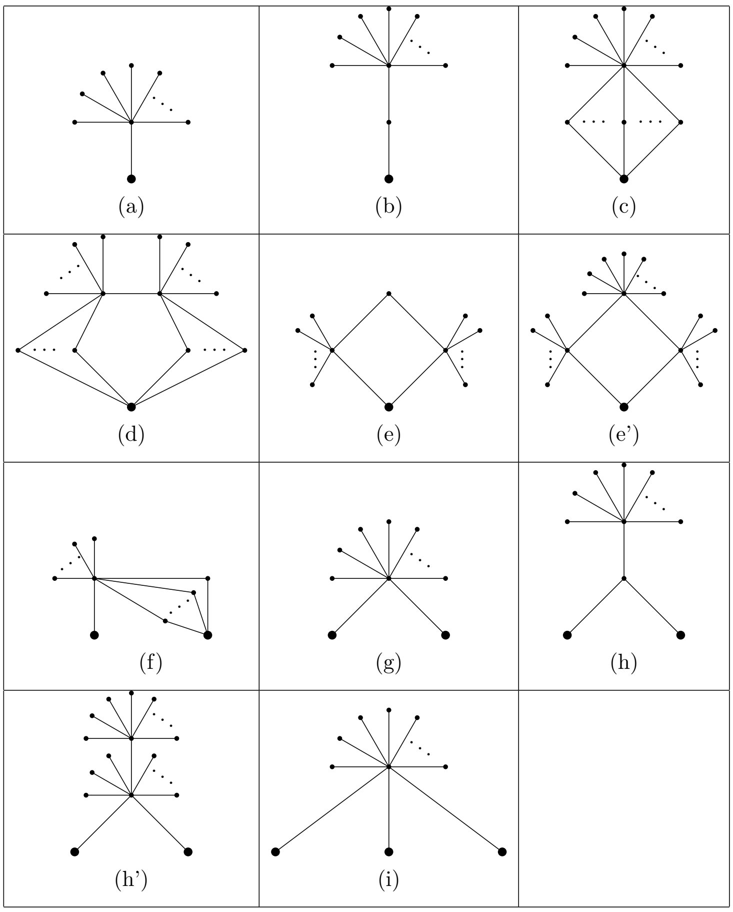
Using the four criteria above, one can deduce that the only possible non-trivial components of each type, referenced from the figure below, are as follows:

(Case: T_1) The non-trivial admissible sequences (up to symmetry) are $(2,6),(3,5)$ and $(4,4)$. The only component type possible for sequence $(2,6)$ is (a) in the figure below. A component type with sequence $(3,5)$ may be (a), (b) or (c). Finally, a component type with sequence $(4,4)$ may be any of (a)-(e).

(Case: T_2) In this case, the possible non-trivial admissible sequences are $(2,4,2)$, $(2,3,3)$, and $(3,2,3)$. The component types with sequence $(2,4,2)$ are (g) and a single edge with one attachment point on each vertex (notice this is a special case of (f) with no pendent edges and only a single path between the two base vertices). The component types with sequence $(2,3,3)$, are (f) and (g). Finally the comopnent types possible with sequence $(3,2,3)$ are (g) and (h).

(Case: T_3) For three attachment points, the only non-trivial sequence possible is $(2,2,2,2)$. The only possible non-trivial component type (with sequence $(2,2,2,2)$) is (i).

(Case: T_4) Finally, for four attachment points, the only sequence possible is $(1,2,2,2,1)$ which is trivial, and thus only the trivial component type is attainable with four attachment points.



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