Degree Ramsey numbers of closed blowups of trees

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Abstract

The *s-color degree Ramsey number* of a graph G, denoted $R_\Delta(G;s)$, is $\min\{\Delta(H)\colon\ H \overset{s}{\to} G\}$, where $H \overset{s}{\to} G$ means that every s-edge-coloring of H contains a monochromatic copy of G. The *closed k-blowup* of a graph is obtained by replacing every vertex with a clique of size k and every edge with a complete bipartite graph where both partite sets have size k; we say that G is a *closed blowup* of H if G is the closed k-blowup of H for some k. We prove that there is a function f such that $R_\Delta(G;s) \le f(\Delta(G),s)$ when G is a closed blowup of a tree.

1 Introduction

When G is a graph, we use $V(G)$ to denote the vertex set of G and $E(G)$ to denote the edge set. The degree of a vertex u in G is denoted by $d(u)$. Given graphs H and G, we write $H \overset{s}{\to} G$ if every s-edge-coloring of H contains a monochromatic copy of G. For graphs, Ramsey's Theorem implies that for every graph G and each positive integer s, there is a graph H such that $H \overset{s}{\to} G$. When $H \overset{s}{\to} G$, we call G the *target graph* and H a *Ramsey host* for G.

The main goal of graph-based Ramsey theory is to understand the relation $H \overset{s}{\to} G$. Typically, a target graph G is fixed and one seeks a Ramsey host for G that has a desired property or is extremal with respect to a certain parameter. The *Ramsey number* of a graph G, denoted $R(G;s)$, is $\min\{|V(H)|\colon\ H \overset{s}{\to} G\}$. Chvátal, Rödl, Szemerédi, and Trotter [4] proved that for each k, there is a constant c_k such that $R(G;2) \le c_k|V(G)|$ whenever G has maximum degree at most k. In other words, the Ramsey numbers of bounded degree graphs grow only linearly with the number of vertices, in marked contrast to the exponential growth that occurs when the bounded degree condition is omitted. Several groups generalized this result to multicolored hypergraphs (see [5] and [6]).

The *size Ramsey number* of G, denoted $R'(G;s)$, is $\min\{|E(H)|\colon\ H \overset{s}{\to} G\}$. Beck [2] proved that for each s, there exists a constant c_s such that $R'(P_n;s) \le c_s n$, where P_n is the path on n vertices. Beck asked whether the size Ramsey numbers of bounded degree graphs also grow linearly in the number of vertices. In addition to paths, Beck's question was answered in the affirmative for

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trees [9] and cycles [10]. However, Rödl and Szemerédi [15] resolved Beck's question in the negative by constructing a family of 3-regular graphs whose size Ramsey numbers grow superlinearly.

We consider a variant of Beck's question where we no longer require our Ramsey hosts to have few edges but we do insist they have bounded degree. The *degree Ramsey number*, denoted $R_\Delta(G;s)$, is $\min\{\Delta(H)\colon H \overset{s}{\to} G\}$, where $\Delta(H)$ denotes the maximum degree in H. The degree Ramsey analogue of Beck's question follows naturally.

Question 1. *Is $R_\Delta(G;s)$ bounded by a function of $\Delta(G)$ and s?*

A family of graphs $\mathcal{G}$ is *R_Δ-bounded* if there is a function $f(d,s)$ such that $R_\Delta(G;s) \le f(\Delta(G),s)$ for every $G \in \mathcal{G}$. Question 1 is then whether or not the family of all graphs is R_Δ-bounded. Paths [1] and cycles [10, 11] are R_Δ-bounded. Extending the Alon et al. argument for paths, Jiang observed that $R_\Delta(T;s) \le 2s(\Delta(T)-1)$ when T is a tree, and this bound is nearly sharp when s and $\Delta(T)$ are large [12]. While we are unable to resolve Question 1, we believe that the family of all graphs is not R_Δ-bounded.

Our work was motivated by a concrete problem in the direction of Question 1. For a graph G, let G^k denote the graph on $V(G)$ where distinct vertices are adjacent if and only if their distance in G is at most k. Is the family of powers of paths R_Δ-bounded? Even the special case of determining whether $R_\Delta(P_n^2;s)$ is bounded by a function of s is not clear.

In this note, we resolve this problem. In fact, we prove more. The *closed k-blowup* of G, denoted $G[k]$, is the graph obtained from G by replacing each vertex in G with a clique of size k and each edge in G with a complete bipartite graph whose partite sets each have size k. We show that the family of closed blowups of trees is R_Δ-bounded. It follows that the family of powers of paths is R_Δ-bounded since P_n^k is a subgraph of $P_{\lceil n/k\rceil}[k]$ and $\Delta(P_{\lceil n/k\rceil}[k]) < \frac{3}{2}\Delta(P_n^k)$ when n is large in terms of k.

One interesting test case for Question 1 is the family of grids $P_n \,\square\, P_n$, where $G \,\square\, H$ is the graph on $V(G)\times V(H)$ with $(u_1,v_1)(u_2,v_2) \in E(G \,\square\, H)$ if and only if $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$. It is not known whether the family of grids is R_Δ-bounded.

In addition to minimizing $|V(H)|$, $|E(H)|$, and $\Delta(H)$, several researchers have sought Ramsey hosts H that are extremal with respect chromatic number [3] and clique number [8, 13, 14]. The former reference also provides exact results on the degree Ramsey numbers of complete graphs and stars; in particular, $R_\Delta(K_n;s) = R(K_n;s) - 1$.

2 Construction

A graph is d-regular if every vertex has degree d, and the *girth* of a graph is the minimum number of vertices in a cycle. Erdős and Sachs [7] proved that for every d and g, there is a d-regular graph with girth g. Alon, Ding, Oporowski, and Vertigan [1] observed that if H has girth at least n and average degree at least $2s$, then $H \overset{s}{\to} P_n$, where P_n is the path on n vertices. Jiang [11] noted that their argument extends to the case that the target graph is a tree. We include the short proof for completeness.

Lemma 2. *If T is a tree with $|V(T)| \geq 3$ and H is a graph with average degree at least $2s(\Delta(T)-1)$ and girth at least $|V(T)|$, then $H \overset{s}{\to} T$.*

Proof. Consider an s-edge-coloring of H and let $n = |V(H)|$. Since H has average degree at least $2s(\Delta(T)-1)$, we have $|E(H)| \geq ns(\Delta(T)-1)$ and hence some color is used on at least $n(\Delta(T)-1)$ edges. Let H_0 be a monochromatic subgraph of H with at least $ns(\Delta(T)-1)$ edges. It follows that H_0 contains a subgraph H_1 with $\delta(H_1) \geq \Delta(T)$. Indeed, if every subgraph of H_0 had a vertex u with $d(u) \leq \Delta(T)-1$, then iteratively deleting vertices of minimum degree would yield $|E(H_0)| \leq (n-1)(\Delta(T)-1) < n(\Delta(T)-1)$, a contradiction. Let H_1 be a subgraph of H_0 with $\delta(H_1) \geq \Delta(T)$. Since H_1 has minimum degree at least $\Delta(T)$ and girth at least $|V(T)|$, a well known greedy embedding strategy finds T as a subgraph of H_1. Hence $H \overset{s}{\to} T$. □

Hence, for each tree T with at least 3 vertices, Lemma 2 implies that $R_\Delta(T;s) \leq 2s(\Delta(T)-1)$. Our main theorem generalizes this bound to the case where the target graph is a closed blowup of a tree. Let $[n] = \{1,\ldots,n\}$, and, when S is a set, let $\binom{S}{k}$ be the set of subsets of S of size k. We will need the well known strengthening of the Erdős and Sachs result that for every d and g, there is a bipartite d-regular graph with girth at least g.

Theorem 3. *Let k and s be integers with $k \geq 2$ and $s \geq 1$, and let $r = R(K_{2k};s)$. If T is a tree with $|V(T)| \geq 3$, then $R_\Delta(T[k];s) \leq (r-1)\left(2\binom{r}{2k}(\Delta(T)-1)\right)^{\binom{r-1}{2k-1}}$.*

Proof. Let $t = \binom{r}{2k}$, $t' = \binom{r-1}{2k-1}$, $d = 2t(\Delta(T)-1)$, and let B be a d-regular (X,Y)-bigraph with girth at least $|V(T)|$. To construct a Ramsey host for $T[k]$, we first use B to construct an r-uniform, r-partite hypergraph F. The partite sets of F are $Z_1,\ldots,Z_r$. The vertices in Z_j are certain t-tuples of elements in $V(B) \cup E(B)$, indexed by $\binom{[r]}{2k}$. When w is such a tuple and $A \in \binom{[r]}{2k}$, we use w_A to denote the A-value of w. For each $A \in \binom{[r]}{2k}$, let A^- be the k smallest integers in A and A^+ be the k largest integers in A. The partite set Z_j consists of all tuples w such that for each coordinate A, the A-value of w belongs to X, Y, or $E(B)$ according to whether $j \in A^-$, $j \in A^+$, or $j \notin A$, respectively. It remains to specify the edges of F.

In the following, we use $w_{j,A}$ to denote the A-value of a vertex $w_j \in Z_j$. When $u \in X$, $v \in Y$, and $uv \in E(B)$, we say that the edge uv *satisfies the A-coordinate* of $w_j \in Z_j$ if

$$w_{j,A} = \begin{cases} u & \text{if } j \in A^- \\ v & \text{if } j \in A^+ \\ uv & \text{if } j \notin A. \end{cases}$$

Consider $w_1,\ldots,w_r$ with $w_j \in Z_j$ for each j. We let $w_1 \ldots w_r \in E(F)$ if and only if for each coordinate $A \in \binom{[r]}{2k}$, there is some edge in B that simultaneously satisfies the A-coordinate of each vertex in $\{w_1,\ldots,w_r\}$. We obtain our host graph H from F by replacing each edge in F with an r-clique in H. Consequently, $w_i \in Z_i$ and $w_j \in Z_j$ are adjacent in H if and only if for each $A \in \binom{[r]}{2k}$, there is an edge in B that satisfies the A-coordinates of w_i and w_j.

To motivate our construction, we note that H contains many copies of $B[k]$, the closed k-blowup of B. For each coordinate $A \in \binom{[r]}{2k}$ and each function $h\colon \left(\binom{[r]}{2k} - \{A\}\right) \to E(B)$, we obtain a copy of $B[k]$ in H. The copy of $B[k]$ in H is specified by the function $g_{A,h}\colon V(B) \to \binom{V(H)}{k}$, defined as follows:

$$g_{A,h}(u) = \{w\colon w_A = u \text{ and } \forall A' \in \tbinom{[r]}{2k} \text{ if } A' \neq A, \text{ then } h(A') \text{ satisfies the } A'\text{-coordinate of } w\}.$$

Note that at most one vertex in each partite set is a candidate for inclusion in $g_{A,h}(u)$. Moreover, the condition $w_A = u$ requires that $u \in X$ and $j \in A^-$ or that $u \in Y$ and $j \in A^+$. Since $|A^-| = |A^+| = k$, always $|g_{A,h}(u)| = k$. Finally, if $uv \in E(B)$ with $u \in X$ and $v \in Y$, it follows from the definition of F that $g_{A,h}(u) \cup g_{A,h}(v)$ is contained in the edge in F where uv satisfies the A-coordinate and the other coordinates are satisfied by the corresponding images of h.

Let $m = |E(B)|$. Since each edge in F is determined by selecting for each $A \in \binom{[r]}{2k}$ an edge in B to satisfy the A-coordinate, we have that $|E(F)| = m^t$. It remains to show that $\Delta(H) \le (r-1)d^{t'}$ and that $H \xrightarrow{s} T[k]$. We claim that F is $d^{t'}$-regular. Consider a vertex $w_j \in Z_j$. Indeed, if w_j belongs to an edge $e \in E(F)$, then each coordinate is satisfied by some edge in B. If $j \notin A$, then $w_{j,A} \in E(B)$ and the A-value of all other vertices is forced. If $j \in A$ and $w_{j,A} = u$, then the A-value of all other vertices is determined by selecting an edge in B to satisfy the A-coordinate, which must be incident to u. Since u is incident to d edges in B and t' of the coordinates in $\binom{[r]}{2k}$ contain j, the claim follows. Moreover, since F is r-uniform, replacing each edge in F with a clique increases the degree at each vertex by at most a factor of $r-1$, and therefore $\Delta(H) \le (r-1)d^{t'}$.

Finally, we show that $H \xrightarrow{s} T[k]$. Consider an s-edge-coloring of H and let e be an edge in F. Since e becomes an r-clique in H and $r = R(K_{2k}; s)$, there is a monochromatic $2k$-clique contained in e. For each $e \in E(F)$, choose $S_e \in \binom{[r]}{2k}$ so that $\{w \in e\colon w \in Z_j \text{ for some } j \in S_e\}$ is a monochromatic $2k$-clique in H. Hence, there exists a coordinate $A \in \binom{[r]}{2k}$ such that at least m^t/t edges $e \in E(F)$ have $S_e = A$; in the following, fix such a coordinate A. The *signature* of an edge $e \in E(F)$ with $S_e = A$ is the function $h\colon \left(\binom{[r]}{2k} - A\right) \to E(B)$ that records, for every other coordinate A' besides A, the edge in B that satisfies A'. Since there are m^{t-1} signatures, some signature is common to at least $\frac{m^t}{t} \cdot \frac{1}{m^{t-1}}$ edges in F. Fix such a signature h. Let F' be the subhypergraph of F consisting of all edges $e \in E(F)$ such that $S_e = A$ and the signature of e is h. Note that F' has at least m/t edges.

We use F' to obtain a subgraph $B^\star$ of B. For each edge e in F', let $e^\star$ be the edge in B that satisfies the A-coordinate. Note that the map $e \mapsto e^\star$ is injective since the signature of each edge in F' is h. Let $B^\star$ be the subgraph of B with edge set $\{e^\star\colon e \in E(F')\}$. We associate each vertex u in B with the k-clique $g_{A,h}(u)$ in H. Note that each edge in $B^\star$ corresponds to a monochromatic $2k$-clique in H. Since B is d-regular with m edges and $B^\star$ has at least m/t edges, it follows that the average degree of $B^\star$ is at least d/t. Since $d/t = 2(\Delta(T) - 1)$, an application of Lemma 2 with $s = 1$ implies that T is a subgraph of $B^\star$. The copy of T in $B^\star$ maps via $g_{A,h}$ to a copy of $T[k]$ in H in which each $2k$-clique corresponding to an edge in T is monochromatic. Since $k \ge 2$ and T is connected, it follows that the copy of $T[k]$ is monochromatic. □

We make no attempt to optimize the bound in Theorem 3. Note that the argument of Theorem 3 also applies in the hypergraph setting. The complete q-uniform n-vertex hypergraph is denoted by $K_n^{(q)}$. Let k and q be integers with $k \ge q$, let s be a positive integer, and let $r = R(K_{2k}^{(q)}; s)$. Let G be the q-uniform k-blowup of a tree T on at least 3 vertices obtained by replacing each vertex u

in T with a set S_u of k vertices and replacing each edge uv in T with a copy of $K_{2k}^{(q)}$ on $S_u \cup S_v$. Repeating the construction above (except that each edge of F is replaced with a copy of $K_r^{(q)}$) yields a q-uniform hypergraph H with $H \xrightarrow{s} G$ and $\Delta(H) \leq \binom{r-1}{q-1}\left(2\binom{r}{2k}(\Delta(T)-1)\right)^{\binom{r-1}{2k-1}}$.

Theorem 3 constructs a Ramsey host for the closed k-blowup of a tree T using a Ramsey host for T. It is natural to ask whether such a construction is possible in general. Is it true that the family of closed blowups of $\mathcal{F}$ is R_Δ-bounded whenever $\mathcal{F}$ is R_Δ-bounded? The analogous statement for *open blowups*, where vertices are replaced with independent sets and edges are replaced with complete bipartite graphs, holds [11].

It is also of interest to find larger R_Δ-bounded graph families. For example, is the family of planar graphs R_Δ-bounded? Since the grids $P_n \square P_n$ are planar, this question seems challenging. However, Theorem 3 implies that the family of outerplanar graphs is R_Δ-bounded. Indeed, if G is a 2-connected outerplanar graph with maximum degree d, then for every edge uv on the outer cycle of G, we have that G is a subgraph of a closed $(2d)$-blowup of a rooted tree T in which each vertex has at most $2d-3$ children and the root of T expands to contain the image of u, v, and their neighbors. This is proved by induction on $|V(G)|$; the neighbors of u and v divide $G-u-v$ into $(d(u)-2)+(d(v)-2)+1$ pieces which may be treated inductively. The claim then follows from the fact that every outerplanar graph with maximum degree d is a subgraph of a 2-connected outerplanar graph with maximum degree at most $d+2$.

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