Edge-disjoint rainbow spanning trees in complete graphs

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Abstract

Let G be an edge-colored copy of K_n, where each color appears on at most $n/2$ edges (the edge-coloring is not necessarily proper). A rainbow spanning tree is a spanning tree of G where each edge has a different color. Brualdi and Hollingsworth [4] conjectured that every properly edge-colored K_n ($n \geq 6$ and even) using exactly $n-1$ colors has $n/2$ edge-disjoint rainbow spanning trees, and they proved there are at least two edge-disjoint rainbow spanning trees. Kaneko, Kano, and Suzuki [13] strengthened the conjecture to include any proper edge-coloring of K_n, and they proved there are at least three edge-disjoint rainbow spanning trees. Akbari and Alipouri [1] showed that each K_n that is edge-colored such that no color appears more than $n/2$ times contains at least two rainbow spanning trees.

We prove that if $n \geq 1,000,000$ then an edge-colored K_n, where each color appears on at most $n/2$ edges, contains at least $\lfloor n/(1000 \log n) \rfloor$ edge-disjoint rainbow spanning trees.

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1 Introduction

Let G be an edge-colored copy of K_n, where each color appears on at most $n/2$ edges (the edge-coloring is not necessarily proper). A rainbow spanning tree is a spanning tree of G such that each edge has a different color. Brualdi and Hollingsworth [4] conjectured that every properly edge-colored K_n ($n \geq 6$ and even) where each color class is a perfect matching has a decomposition of the edges of K_n into $n/2$ edge-disjoint rainbow spanning trees. They proved there are at least two edge-disjoint rainbow spanning trees in such an edge-colored K_n. Kaneko, Kano, and Suzuki [13] strengthened the conjecture to say that for any proper edge-coloring of K_n ($n \geq 6$) contains at least $\lfloor n/2 \rfloor$ edge-disjoint rainbow spanning trees, and they proved there are at least three edge-disjoint rainbow spanning trees. Akbari and Alipour [1] showed that each K_n that is an edge-colored such that no color appears more than $n/2$ times contains at least two rainbow spanning trees.

Our main result is

Theorem 1. *Let G be an edge-colored copy of K_n, where each color appears on at most $n/2$ edges and $n \geq 1,000,000$. The graph G contains at least $\lfloor n/(1000 \log n) \rfloor$ edge-disjoint rainbow spanning trees.*

The strategy of the proof of Theorem 1 is to randomly construct $\lfloor n/(1000 \log n) \rfloor$ edge-disjoint subgraphs of G such that with high probability each subgraph has a rainbow spanning tree. This result is the best known for the conjecture by Kaneko, Kano, and Suzuki. Horn [12] has shown that if the edge-coloring is a

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proper coloring where each color class is a perfect matching then there are at least ϵn rainbow spanning trees for some positive constant ϵ, which is the best known result for the conjecture by Brualdi and Hollingsworth.

There have been many results in finding rainbow subgraphs in edge-colored graphs; Kano and Li [14] surveyed results and conjecture on monochromatic and rainbow (also called heterochromatic) subgraphs of an edge-colored graph. Related work includes Brualdi and Hollingsworth [5] finding rainbow spanning trees and forests in edge-colored complete bipartite graphs, and Constantine [8] showing that for certain values of n there exists a proper coloring of K_n such that the edges of K_n decompose into isomorphic rainbow spanning trees.

The existence of rainbow cycles has also been studied. Albert, Frieze, and Reed [2] showed that for an edge-colored K_n where each color appears at most $\lceil cn \rceil$ times then there is a rainbow hamiltonian cycle if $c < 1/64$ (Rue (see [11]) provided a correction to the constant). Frieze and Krivelevich [11] proved that there exists a c such that if each color appears at most $\lceil cn \rceil$ times then there are rainbow cycles of all lengths.

This paper is organized as follows. Section 2 includes definitions and results used throughout the paper. Sections 3, 4, and 5 contain lemmas describing properties of the random subgraphs we generate. The final section provides the proof of our main result.

2 Definitions

First we establish some notation that we will use throughout the paper. Let G be a graph and $S \subseteq V(G)$. Let $G[S]$ denote the induced subgraph of G on the vertex set S. Let $[S,\overline{S}]_G$ be the set of edges between S and $\overline{S}$ in G. For natural numbers q and k, $[q]$ represents the set $\{1,\dots,q\}$, and $\binom{[q]}{k}$ is the collection of all k-subsets of $[q]$. Throughout the paper the logarithm function used has base e. One inequality that we will use often is the union sum bound which states that for events $A_1,\dots,A_r$

$$\mathbb{P}\left[\bigcup_{i=1}^{r} A_i\right] \leq \sum_{i=1}^{r} \mathbb{P}\left[A_i\right].$$

Throughout the rest of the paper let G be an edge-colored copy of K_n, where the set of edges of each color has size at most $n/2$, and $n \geq 1{,}000{,}000$. We assume G is colored with q colors, where $n-1 \leq q \leq \binom{n}{2}$. Let C_j be the set of edges of color j in G. Define $c_j = |C_j|$, and without loss of generality assume $c_1 \geq c_2 \geq \cdots \geq c_q$. Note that $1 \leq c_j \leq n/2$ for all j.

Let $t = \lfloor n/(C\log n) \rfloor$ where $C = 1000$. Note that we have not optimized the constant C, and it can be slightly improved at the cost of more calculation. Since $\frac{n}{C\log n} - 1 \leq t \leq \frac{n}{C\log n}$ we have

$$\frac{-1}{t} \leq \frac{-C\log n}{n} \quad \text{and} \quad \frac{C\log n}{n} \leq \frac{1}{t} \leq \left(\frac{n}{n - C\log n}\right)\frac{C\log n}{n}. \tag{$*$}$$

We will frequently use these bounds on t.

We construct edge-disjoint subgraphs $G_1,\dots,G_t$ of G in the following way: independently and uniformly select each edge of G to be in G_i with probability $1/t$. Each G_i (considered as an uncolored graph) is distributed as an Erdős-Rényi random graph $G(n, 1/t)$. Note that the subgraphs are not independent. We will show that with high probability each of the subgraphs $G_1,\dots,G_t$ simultaneously contain a rainbow spanning tree.

To prove that a graph has a rainbow spanning tree we will use Theorem 2 below that gives necessary and sufficient conditions for the existence of a rainbow spanning tree. Broersma and Li [3] showed that determining the largest rainbow spanning forest of H can be solved by applying the Matroid Intersection Theorem [10] (see Schrijver [16, p. 700]), to the graphic matroid and the partition matroid on the edge set of H defined by the color classes. Schrijver [16] translated the conditions of the Matroid Intersection Theorem into necessary and sufficient conditions for the existence of a rainbow spanning tree. Suzuki [17] and Carraher and Hartke [6] gave graph-theoretical proofs of this same theorem.

Theorem 2. *A graph G has a rainbow spanning tree if and only if, for every partition π of $V(G)$, at least $s-1$ different colors are represented between the parts of π, where s is the number of parts of π.*

We show that for every partition π of $V(G)$ into s parts there are at least $s-1$ colors between the parts for each G_i. Sections 3, 4 and 5 describe properties of the subgraphs $G_1, \ldots, G_t$ for certain partitions π of $V(G)$ into s parts. Many of our proofs use the following variant of Chernoff's inequality [7], frequently attributed to Bernstein (see [9]).

Lemma 3 (Bernstein's Inequality)**.** *Suppose X_i are independently identically distributed Bernoulli random variables, and $X = \sum X_i$. Then*

$$\mathbb{P}\left[X \geq \mathbb{E}[X] + \lambda\right] \leq \exp\left(-\frac{\lambda^2}{2(\mathbb{E}[X] + \lambda/3)}\right)$$

and

$$\mathbb{P}\left[X \leq \mathbb{E}[X] - \lambda\right] \leq \exp\left(-\frac{\lambda^2}{2\mathbb{E}[X]}\right).$$

In several places in the paper we use Jensen's inequality.

Lemma 4 (Jensen's Inequality (see [18]))**.** *Let $f(x)$ be a real-valued convex function defined on an interval $I = [a, b]$. If $x_1, \ldots, x_n \in I$ and $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$, then*

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i).$$

We also make use of the following upper bound for binomial coefficients $\binom{n}{k} \leq n^k$.

3 Partitions with n or $n-1$ parts

In this section we show that a partition π of $V(G)$ into n or $n-1$ parts has enough colors between the parts. Since color classes can have small size, there might not be any edges of a given color in a subgraph G_i. Therefore, we group small color classes together to form larger pseudocolor classes. Recall that c_j is the size of the color class C_j, and $c_1 \geq c_2 \geq \cdots \geq c_q$. Define the pseudocolor classes $D_1, \ldots, D_{n-1}$ of G recursively as follows:

$$D_k = \left(\bigcup_{j=1}^{\ell} C_j\right) \setminus \left(\bigcup_{i=1}^{k-1} D_i\right),$$

where ℓ is the smallest integer such that $\left|\left(\bigcup_{j=1}^{\ell} C_j\right) \setminus \left(\bigcup_{i=1}^{k-1} D_i\right)\right| \geq n/4$. Note that the $n-1$ pseudocolor classes might not contain all the edges of G.

Lemma 5. *Each of the $n-1$ pseudocolor classes $D_1, \ldots, D_{n-1}$ have size at least $n/4$ and at most $n/2$.*

Proof. We prove this statement by induction on k. Consider the pseudocolor class D_k, for $1 \leq k \leq n-1$. Since each of the pseudocolor classes $D_1 \ldots, D_{k-1}$ has size at most $n/2$, there are at least $\frac{n}{2}(n-k)$ edges not in $\bigcup_{i=1}^{k-1} D_i$. Therefore there exist ℓ' and ℓ such that $D_k = \bigcup_{i=\ell'}^{\ell} C_i$, where $|D_k| = \sum_{i=\ell'}^{\ell} c_i \geq n/4$.

If $\ell' = \ell$ then $|D_k| = |C_\ell| \leq n/2$. Otherwise, we know $c_\ell \leq c_{\ell-1} \leq c_{\ell'} \leq n/4$. So,

$$|D_k| = \sum_{i=\ell'}^{\ell-1} c_i + c_\ell \leq \frac{n}{4} + c_\ell \leq \frac{n}{4} + \frac{n}{4} = \frac{n}{2},$$

which proves that the pseudocolor class D_k has size at most $\frac{n}{2}$. □

Lemma 6. *For a fixed subgraph G_i and pseudocolor class D_j,*

$$\mathbb{P}\left[|E(G_i) \cap D_j| \le \frac{|D_j|}{t} - \sqrt{3\frac{n}{t}\log n}\right] \le \frac{1}{n^3}.$$

As a consequence, with probability at least $1 - \frac{1}{n}$ every subgraph G_i has at least one edge from each of the pseudocolor classes $D_1, \ldots, D_{n-1}$.

Proof. Fix a subgraph G_i and a pseudocolor class D_j. The expected number of edges in G_i from the pseudocolor class is $\frac{|D_j|}{t}$. By Bernstein's inequality where $\lambda = \sqrt{3\frac{n}{t}\log n}$, we have

$$\begin{aligned}\mathbb{P}\left[|E(G_i) \cap D_j| \le \frac{|D_j|}{t} - \sqrt{3\frac{n}{t}\log n}\right] &\le \exp\left(\frac{-3\frac{n}{t}\log n}{2\frac{|D_j|}{t}}\right) \\ &\le \exp\left(\frac{-3n\log n}{2\frac{n}{2}}\right) = \frac{1}{n^3}.\end{aligned}$$

Since $|D_j| \ge n/4$, $n \ge 1,000$, and $C \ge 50$,

$$\frac{|D_j|}{t} - \sqrt{3\frac{n}{t}\log n} \ge \frac{n}{4t} - \sqrt{3\frac{n}{t}\log n} \ge 1.$$

The second statement follows from the previous inequalities by using the union sum bound for the $n-1$ pseudocolor classes and t subgraphs and recalling that $t < n$. □

Lemma 6 shows that if we consider a partition π of $V(G)$ into s parts, where $s = n$ there must be at least $n-1$ colors in G_i between the parts of π. In the case when the partition has $s = n-1$ parts there is at most one edge inside the parts of π, so there are at least $n-2$ colors in G_i between the parts of π.

4 Partitions where $\left(1 - \frac{14}{\sqrt{C}}\right)n \le s \le n-2$

In this section we consider partitions π of $V(G)$ into s parts where $\left(1 - \frac{14}{\sqrt{C}}\right)n \le s \le n-2$. First, we introduce a new function that will help with our calculations. The function f will be used to bound the probability that $q-(s-2)$ colors do not appear between the parts of π in G_i.

Lemma 7. *For an integer ℓ and real numbers $c_1, \ldots, c_q$, define*

$$f(c_1, \ldots, c_q; \ell) = \sum_{I \in \binom{[q]}{q-\ell}} \exp\left(-\frac{1}{t}\sum_{j \in I} c_j\right).$$

If $1 \le c_j \le \frac{n}{2}$ for each j, $\sum_{i=1}^{q} c_j = \binom{n}{2}$, and $\frac{n}{2} \le \ell \le n-4$, then

$$f(c_1, \ldots, c_q; \ell) \le \exp\left(-\frac{49C}{200}(n-\ell)\log n\right).$$

Proof. For convenience we define $w(I) = \sum_{j \in I} c_j$ for a subset $I \subseteq [q]$.

Claim 1.

$$f(c_1, \ldots, c_q; \ell) \le f(\underbrace{1, 1, \ldots, 1}_{k-1 \text{ times}}, x^*, \underbrace{\frac{n}{2}, \ldots, \frac{n}{2}}_{q-k \text{ times}}; \ell),$$

where $1 \le x^* < \frac{n}{2}$, and where k and x^* are so that $(k-1) + (q-k)\frac{n}{2} + x^* = \binom{n}{2}$.

Proof of Claim 1. Since $f(c_1, \ldots, c_q; \ell)$ is a symmetric function in the c_j's, it suffices to show that when $c_2 \geq c_1$,

$$f(c_1, c_2, \ldots, c_q; \ell) \leq f(c_1 - \epsilon, c_2 + \epsilon, \ldots, c_q; \ell),$$

where $\epsilon = \min\{c_1 - 1, \frac{n}{2} - c_2\}$.

$$\begin{aligned} f(c_1, c_2, \ldots, c_q; \ell) = &\sum_{I \in \binom{[q]\setminus\{1,2\}}{q-\ell}} \exp\left(-\frac{w(I)}{t}\right) + \sum_{I \in \binom{[q]\setminus\{1,2\}}{q-\ell-2}} \exp\left(-\frac{c_1}{t} - \frac{c_2}{t} - \frac{w(I)}{t}\right) \\ &+ \sum_{I \in \binom{[q]\setminus\{1,2\}}{q-\ell-1}} \left(\exp\left(-\frac{c_1}{t} - \frac{w(I)}{t}\right) + \exp\left(-\frac{c_2}{t} - \frac{w(I)}{t}\right)\right) \end{aligned}$$

The first two summations are unchanged in $f(c_1 - \epsilon, c_2 + \epsilon, \ldots, c_q; \ell)$, and hence it suffices to show that for every $I \in \binom{[q]\setminus\{1,2\}}{\ell-1}$,

$$\begin{aligned} &\exp\left(-\frac{c_1}{t} - \frac{w(I)}{t}\right) + \exp\left(-\frac{c_2}{t} - \frac{w(I)}{t}\right) \\ &\qquad \leq \exp\left(-\frac{(c_1 - \epsilon)}{t} - \frac{w(I)}{t}\right) + \exp\left(-\frac{(c_2 + \epsilon)}{t} - \frac{w(I)}{t}\right). \end{aligned}$$

This follows immediately by Jensen's inequality and the convexity of $\exp(\alpha x + \beta)$ as a function in x. □

Claim 2.

$$f(\underbrace{1, 1, \ldots, 1}_{k-1 \text{ times}}, x^*, \underbrace{\frac{n}{2}, \ldots, \frac{n}{2}}_{q-k \text{ times}}; \ell) \leq f(\underbrace{1, \ldots, 1}_{k \text{ times}}, \underbrace{\frac{n}{2}, \ldots, \frac{n}{2}}_{q-k \text{ times}}; \ell),$$

where $\frac{n(n-2)}{2} \leq k + (q-k)\frac{n}{2} \leq \binom{n}{2}$.

Note that since $\binom{n}{2} = k - 1 + x^* + (q-k)\frac{n}{2}$, $x^* < \frac{n}{2}$ and so $x^* - 1 < \frac{n}{2}$, we have $k + (q-k)\frac{n}{2} = \binom{n}{2} - (x^* - 1) > \frac{n(n-1)}{2} - \frac{n}{2} = \frac{n(n-2)}{2}$.

Proof of Claim 2. The function f is decreasing in each c_j, and in particular c_k. □

Next we consider $\sum_{I \in \binom{[q]}{q-\ell}} \exp\left(-\frac{1}{t} w(I)\right)$ and sum up subsets by intersection of the number of ones that appear. Let r be the number of ones. So we have

$$\begin{aligned} f(\underbrace{1, \ldots, 1}_{k \text{ times}}, \underbrace{\frac{n}{2}, \ldots, \frac{n}{2}}_{q-k \text{ times}}; \ell) &= \sum_{I \in \binom{[q]}{q-\ell}} \exp\left(-\frac{1}{t} w(I)\right) \\ &\leq \sum_{r=\max\{0, \ell-(q-k)\}}^{\min\{\ell, k\}} \binom{k}{r}\binom{q-k}{\ell-r} \exp\left(-\frac{1}{t}\left(\frac{n(n-2)}{2} - (\ell - r)\frac{n}{2} - r\right)\right) \\ &\leq \sum_{r=\max\{0, \ell-(q-k)\}}^{\min\{\ell, k\}} k^r (q-k)^{(q-k)-(\ell-r)} \exp\left(-\frac{1}{t}\left(\frac{n}{2}(n - (\ell - r) - 2) - r\right)\right) \\ &\leq \sum_{r=\max\{0, \ell-(q-k)\}}^{\min\{\ell, k\}} \exp\left((q - k - \ell + 2r)\log n - \frac{1}{t}\left(\frac{n}{2}(n - \ell + r - 2) - r\right)\right) \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{r=\max\{0,\ell-(q-k)\}}^{\min\{\ell,k\}} \exp\left(\log n\left(q-k-\ell+2r-\frac{C}{2}(n-\ell+r-2)+\frac{Cr}{n}\right)\right) \quad \text{by } (*)\\
&\le n\exp\left(\log n\left((n-\ell)\left(1-\frac{C}{2}\right)+r\left(2-\frac{C}{2}+\frac{C}{n}\right)+C\right)\right)\\
&\le \exp\left(\log n\left((n-\ell)\left(1-\frac{C}{2}\right)+C+1\right)\right).
\end{aligned}
$$

Since $n-\ell \ge 4$ and $C \ge 250$, we have

$$
1+\frac{1}{n-\ell} \le \frac{C}{200} \le C\left(\frac{1}{2}-\frac{1}{n-\ell}-\frac{49}{200}\right).
$$

Thus the sum above is bounded by

$$
\exp\left(-\frac{49C}{200}(n-\ell)\log n\right).
$$

□

Lemma 8. *Let* Π *be the set of partitions of* $V(G)$ *into* s *parts, where* $\left(1-\frac{14}{\sqrt{C}}\right)n \le s \le n-2$. *For a partition* $\pi \in \Pi$, *let* $\mathcal{B}_{\pi,i}$ *be the event that there are less than* $s-1$ *colors between the parts of* π *in* G_i. *Then*

$$
\mathbb{P}\left[\bigcup_{i=1}^{t}\bigcup_{\pi\in\Pi}\mathcal{B}_{\pi,i}\right] \le \frac{1}{n}.
$$

Proof. Fix a subgraph G_i and a partition $\pi \in \Pi$. Recall that $C_1, \ldots, C_q$ are the color classes of G with sizes $c_1, \ldots, c_q$, respectively. Let $I_{\pi,i}$ be the set of colors that do not appear on edges of G_i between the parts of π.

The total number of edges in G that have a color indexed by $I_{\pi,i}$ is $\sum_{j\in I_{\pi,i}} c_j$. By convexity of $\binom{x}{2}$, there are at most $\binom{n-s+1}{2}$ edges inside the parts of π. Note that when event $\mathcal{B}_{\pi,i}$ happens that $|I_{\pi,i}| \ge q-(s-2)$ and if $I_{\pi,i}$ does not have size $q-(s-2)$, then it contains a set $I' \subseteq I_{\pi,i}$ of size $q-(s-2)$, and the event that no edges of G_i between the parts of π have colors in $I_{\pi,i}$ is contained in the event that no edges of G_i between the parts have colors in I'. Thus,

$$
\begin{aligned}
\mathbb{P}\left[\mathcal{B}_{\pi,i}\right] &\le \sum_{I\in\binom{[q]}{q-(s-2)}} \left(1-\frac{1}{t}\right)^{\sum_{j\in I} c_j-\binom{n-s+1}{2}}\\
&\le f(c_1,c_2,\ldots,c_q;s-2)\left(1-\frac{1}{t}\right)^{-\binom{n-s+1}{2}}\\
&\le f(c_1,c_2,\ldots,c_q;s-2)\exp\left(\frac{1}{t}\binom{n-s+1}{2}\right) \qquad \text{since } \left(1-\frac{1}{t}\right)\le e^{-\frac{1}{t}}\\
&\le \exp\left(-\frac{49C}{200}\left(n-(s-2)\right)\log n+\frac{(n-s+1)^2}{2t}\right) \qquad \text{by Lemma 7.}
\end{aligned}
$$

Since $s \ge \left(1-\frac{14}{\sqrt{C}}\right)n$, we know $n-s+1 \le \frac{14n}{\sqrt{C}}+1$. Thus we can bound the previous line by

$$
\begin{aligned}
&\le \exp\left((n-s+1)\left(-\frac{49C}{200}\log n+\frac{1}{2t}\left(\frac{14}{\sqrt{C}}n+1\right)\right)\right)\\
&\le \exp\left((n-s+1)\log n\left(-\frac{49C}{200}+\frac{n}{n-C\log n}\left(\frac{14\sqrt{C}}{2}+\frac{C}{2n}\right)\right)\right) \qquad \text{by } (*).
\end{aligned}
$$

We now perform a union bound over all partitions $\pi \in \Pi$. The number of partitions of $V(G)$ into s nonempty parts is at most

$$\binom{n}{s} s^{n-s} \leq \binom{n}{n-s} n^{n-s} \leq n^{2(n-s)} = \exp(2(n-s)\log n) \leq \exp(2(n-s+1)\log n).$$

Therefore,

$$\mathbb{P}\left[\bigcup_{\substack{\pi\in\Pi \\ \text{with } s \text{ parts}}} \mathcal{B}_{\pi,i}\right] \leq \exp\left((n-s+1)\log n\left(2 - \frac{49C}{200} + \frac{n}{n - C\log n}\left(\frac{14\sqrt{C}}{2} + \frac{C}{2n}\right)\right)\right).$$

Since $C = 1000$ and $n \geq 1,000,000$, we have

$$2 - \frac{49C}{200} + \frac{n}{n - C\log n}\left(\frac{14\sqrt{C}}{2} + \frac{C}{2n}\right) \leq -1,$$

and since $(n-s+1) \geq 3$,

$$\mathbb{P}\left[\bigcup_{\substack{\pi\in\Pi \\ \text{with } s \text{ parts}}} \mathcal{B}_{\pi,i}\right] \leq \exp\left(-3\log n\right) = \frac{1}{n^3}.$$

This gives a bound on the probability for a fixed partition size s. Using the union sum bound over all partition sizes s, where $\left(1 - \frac{14}{\sqrt{C}}\right) n \leq s \leq n-2$, and over all t subgraphs completes the proof. □

This proves when s is large there are enough colors between the parts.

5 Partitions where $2 \leq s \leq \left(1 - \frac{14}{\sqrt{C}}\right) n$

Next, we prove several results that will be used to show there are enough colors in G_i between the parts of the partition when the number of parts is small. Our goal is to show that for a partition π of $V(G)$ into s parts, the number of edges between the parts in G_i is so large that there must be at least $s-1$ colors between the parts.

Lemma 9. *For a fixed subgraph G_i and color j,*

$$\mathbb{P}\left[|E(G_i) \cap C_j| \geq \frac{n}{2t} + 4\sqrt{\frac{n}{t}\log n}\right] \leq \frac{1}{n^4}.$$

As a consequence, with probability at least $1 - \frac{1}{n}$, every color appears at most $\frac{n}{2t} + 4\sqrt{\frac{n}{t}\log n}$ times in every G_i.

Proof. Fix a color j and a subgraph G_i. Order the edges of C_j as $e_1, \ldots, e_{c_j}$. For $1 \leq k \leq c_j$, let X_k be the indicator random variable for the event $e_k \in E(G_i)$. For a color class with size less than $\frac{n}{2}$ we introduce dummy random variables, so we can apply Bernstein's inequality. For $c_j + 1 \leq k \leq n/2$, let X_k be a random variable distributed independently as a Bernoulli random variable with probability $1/t$.

By construction, $|E(G_i)\cap C_j| \le X = \sum_{k=1}^{n/2} X_k$ and $\mathbb{E}[X] = \frac{n}{2t}$. By Bernstein's Inequality where $\lambda = 4\sqrt{\frac{n}{t}\log n}$, we have

$$
\begin{aligned}
\mathbb{P}\left[|E(G_i)\cap C_j| \ge \frac{n}{2t} + 4\sqrt{\frac{n}{t}\log n}\right] &\le \mathbb{P}\left[X \ge \frac{n}{2t} + 4\sqrt{\frac{n}{t}\log n}\right] \\
&\le \exp\left(\frac{-\frac{16n}{t}\log n}{2\left(\frac{n}{2t} + \frac{4}{3}\sqrt{\frac{n}{t}\log n}\right)}\right) \\
&= \exp\left(\frac{-16\log n}{1+\frac{8}{3}\sqrt{\frac{t}{n}\log n}}\right) \\
&\le \exp\left(\frac{-16}{1+\frac{8}{3\sqrt{C}}}\log n\right) \quad \text{since } t \le \frac{n}{C\log n}, \\
&\le \exp\left(\frac{-16}{\frac{11}{3}}\log n\right) \le \left(\frac{1}{n}\right)^{48/11} \le \frac{1}{n^4} \quad \text{since } C \ge 1,
\end{aligned}
$$

which proves the first statement.

The second statement follows from the previous inequality by using the union sum bound for the q color classes and t subgraphs, and recalling that $q < n^2$ and $t < n$. ☐

Lemma 10. *Fix $S \subseteq V(G)$. Let $\mathcal{B}_{S,i}$ be the event*

$$\left|\left[S,\overline{S}\right]_{G_i}\right| \le \frac{|S|(n-|S|)}{t} - \sqrt{\frac{6|S|(n-|S|)}{t}\min\{|S|, n-|S|\}\log n}.$$

Then

$$\mathbb{P}\left[\bigcup_{i=1}^{t}\bigcup_{S\subseteq V(G)}\mathcal{B}_{S,i}\right] \le \frac{4}{n}.$$

Proof. Fix a subgraph G_i and a set of vertices $S \subseteq V(G)$. Let $r = |S|$. The expected number of edges in G_i between S and $\overline{S}$ is $r(n-r)/t$. By Bernstein's inequality with $\lambda = \sqrt{6\frac{r(n-r)}{t}\min\{r,n-r\}\log n}$, we have

$$\mathbb{P}\left[\mathcal{B}_{S,i}\right] \le \exp\left(\frac{-6\frac{r(n-r)}{t}\min\{r,n-r\}\log n}{2\frac{r(n-r)}{t}}\right) = n^{-3\min\{r,n-r\}}.$$

So

$$
\begin{aligned}
\mathbb{P}\left[\bigcup_{S\subseteq V(G)}\mathcal{B}_{S,i}\right] &\le \sum_{r=1}^{n/2}\binom{n}{r}n^{-3r} + \sum_{r=n/2}^{n}\binom{n}{n-r}n^{-3(n-r)} = 2\sum_{r=1}^{n/2}\binom{n}{r}n^{-3r} \\
&\le 2\sum_{r=1}^{n/2}n^{-2r} \le 2n^{-2} + 2\left(\sum_{r=2}^{n/2}n^{-4}\right) \le \frac{2}{n^2} + \frac{2}{n^3} \le \frac{4}{n^2}.
\end{aligned}
$$

Applying the union sum bound for the t subgraphs gives the final statement of the lemma. ☐

The previous lemma gives a lower bound on the number of edges between S and $\overline{S}$. We use this lemma to find a lower bound on the number of edges between the parts for a partition $\pi = \{P_1, \ldots, P_s\}$ of $V(G)$.

Definition 11. *For $x \in [0,n]$, let*

$$f(x) = \frac{x(n-x)}{t} - \sqrt{\frac{6x(n-x)}{t} \min\{x, n-x\} \log n}.$$

If none of the bad events $\mathcal{B}_{S,i}$ from Lemma 10 occur, then the sum $\frac{1}{2}\sum_{\pi=\{P_1,\ldots,P_s\}} f(|P_i|)$, where $\sum_{i=1}^{s} |P_i| = n$, is a lower bound on the number of edges between the parts of the partition π. We bound this sum for all partitions. If $-f(x)$ was convex then we could immediately find a lower bound by using Jensen's inequality in Lemma 4. Since $-f(x)$ is not convex, we bound it with a function that is convex.

Let $h(x)$ be a function with domain $[a,b]$. We say a function h is *concave* if for $x, y \in [a,b]$ and $\lambda \in [0,1]$, then $h(\lambda x + (1-\lambda)y) \geq \lambda h(x) + (1-\lambda)h(y)$. First, we present two basic results about concave functions.

Lemma 12. *Let $h(x)$ be a differentiable function with domain $[a,b]$. Suppose that h is concave on $[z,b]$, where $z \in (a,b)$. Let $\ell(x)$ be the line tangent to h at the point $(z, h(z))$. Then the function*

$$h_1(x) = \begin{cases} \ell(x) & \text{if } a \leq x \leq z, \\ h(x) & \text{if } z < x \leq b \end{cases}$$

is concave.

It is well known (see Proposition 3.10 [15]) that a differentiable function $f(x)$ on an interval $[a,b]$ is concave if and only if $f'(x)$ is weakly decreasing. The function $h_1(x)$ is defined in a way such that $h_1(x)$ is differentiable and the derivative is initially constant and then weakly decreasing and hence this result applies.

Lemma 13. *If h_1 and h_2 are concave functions, then $h(x) = \min\{h_1(x), h_2(x)\}$ is concave.*

The proof for Lemma 13 can be found as Example 2.15 in Peypouquet [15].

We next define several functions that will lead to a concave lower bound for the function f. Define on $[0,n]$ the functions

$$f_1(x) = \frac{x(n-x)}{t} - x\sqrt{\frac{6(n-x)}{t} \log n},$$
$$f_2(x) = \frac{x(n-x)}{t} - (n-x)\sqrt{\frac{6x}{t} \log n}.$$

Note that

$$f(x) = \begin{cases} f_1(x) & 0 \leq x \leq n/2, \\ f_2(x) & n/2 < x \leq n. \end{cases}$$

Let $\ell(x) = f_2'(x)(x - n/2) + f_2(n/2)$ be the tangent line of $f_2(x)$ at the point $\left(\frac{n}{2}, \frac{n^2}{4t} - \frac{n}{2}\sqrt{\frac{3n}{t}\log n}\right)$. Let c be the point such that $f_1(x)$ achieves its maximum value on the interval $[0,n]$. Define

$$f_3(x) = \begin{cases} \ell(x) & 0 \leq x \leq n/2, \\ f_2(x) & n/2 < x \leq n \end{cases}$$

and

$$f_4(x) = \begin{cases} f_1(x) & 0 \leq x \leq c, \\ f_1(c) & c < x \leq n. \end{cases}$$

By Lemma 12 the functions f_3 and f_4 are concave.

On the interval $[0,n]$ define $f_5(x) = \min\{f_3(x), f_4(x)\}$. The function $f_5(x)$ is concave by Lemma 13, where $f(x) \geq f_5(x)$ for all $x \in [0,n]$. Figure 1 shows the functions $f(x)$ and $\ell(x)$ used to create $f_5(x)$.

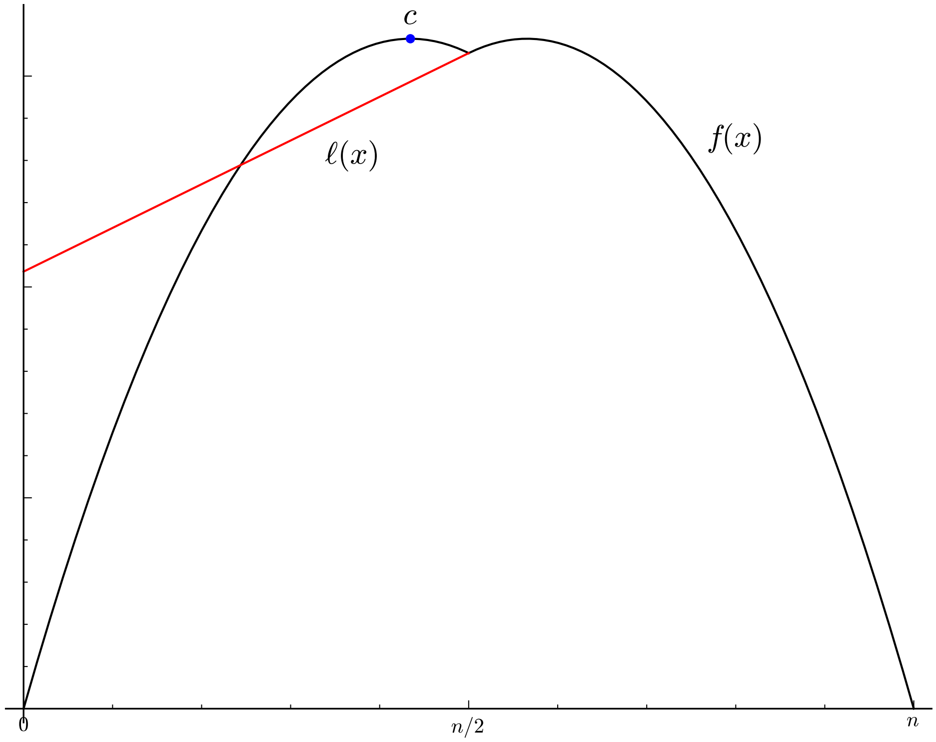

Figure 1: The function $f(x)$, along with the line $\ell(x)$.

Lemma 14. *The sum* $\sum_{i=1}^{s} f(x_i)$*, where* $\sum_{i=1}^{s} x_i = n$ *and* $x_i \geq 1$ *for all* i*, is bounded below by*

$$\sum_{i=1}^{s} f(x_i) \geq (s-1)f(1) + f(n-s+1).$$

Proof. The proof is broken up into two cases based on whether $s \leq n/2+1$, or $s > n/2+1$.

When $s \leq n/2+1$ the function $f(x) \geq f_5(x)$, so $\sum_{i=1}^{s} f(x_i) \geq \sum_{i=1}^{s} f_5(x_i)$. Since the function $f_5(x)$ is concave the sum $\sum_{i=1}^{s} f_5(x)$ is minimized when there is one part of size $n-s+1$ and all the other parts are of size 1. Since $n-s+1 \geq n/2$, we have $f_5(n-s+1) = f(n-s+1)$. Note that $\ell(1) \geq f_1(1)$, which implies $f_5(1) = f(1)$. Thus

$$\sum_{i=1}^{s} f(x_i) \geq \sum_{i=1}^{s} f_5(x_i) \geq (s-1)f_5(1) + f_5(n-s+1) = (s-1)f(1) + f(n-s+1).$$

When $s > n/2+1$, we have $x_i \leq n/2$ for all i. Therefore $f(x_i) = f_1(x_i)$ for all i. Since $f_1(x)$ is concave the sum is minimized when one part has size $n-s+1$ and the rest have size 1. $\square$

Lemma 15. *Let* π *be a partition of the vertices of* G *into* s *parts. Suppose none of the events* $\mathcal{B}_{S,i}$ *from Lemma 10 hold for all* $S \subseteq V(G)$ *and* $1 \leq i \leq t$*. Then in each of the subgraphs* $G_1, \ldots, G_t$*, the number of edges between the parts of* π *is at least*

$$\frac{1}{2}\left((s-1)\left(\frac{n-1}{t} - \sqrt{6(n-1)\frac{\log n}{t}}\right) + \frac{(n-s+1)(s-1)}{t} - (s-1)\sqrt{6(n-s+1)\frac{\log n}{t}}\right)$$

when $s \leq n/2+1$, *and*

$$\frac{1}{2}\left((s-1)\left(\frac{n-1}{t}-\sqrt{6(n-1)\frac{\log n}{t}}\right)+\frac{(n-s+1)(s-1)}{t}-(n-s+1)\sqrt{6(s-1)\frac{\log n}{t}}\right)$$

when $s > n/2+1$.

Proof. If none of the events $\mathcal{B}_{S,i}$ hold then the sum $\frac{1}{2}\sum_{\pi=\{P_1,\ldots,P_s\}} f(x)$ where $\sum_{i=1}^{s}|P_i| = n$ is a lower bound on the number of edges between the parts of π. By Lemma 14 we know this sum is bounded below by $\frac{1}{2}\left((s-1)f(1)+f(n-s+1)\right)$. ☐

Lemma 16. *Let π be a partition of the vertices of G into s parts, where $2 \leq s \leq \left(1-\frac{14}{\sqrt{C}}\right)n$. Suppose none of the events $\mathcal{B}_{S,i}$ from Lemma 10 hold for all $S \subseteq V(G)$ and $1 \leq i \leq t$, and every color appears in each G_i at most $\frac{n}{2t}+4\sqrt{\frac{n}{t}\log n}$ times (as in Lemma 9). Then in each of the subgraphs $G_1,\ldots,G_t$, the number of colors between the parts of π is at least $s-1$.*

Proof. Suppose there exists a subgraph G_i and a partition π into s parts where there are at most $s-2$ colors between the parts in G_i. Then by assumption there are at most

$$(s-2)\left(\frac{n}{2t}+4\sqrt{\frac{n}{t}\log n}\right)$$

edges in G_i between the parts of π. We will show that the number of edges between the parts of π cannot be this small, giving a contradiction.

Suppose $\frac{n}{2}+1 < s \leq \left(1-\frac{14}{\sqrt{C}}\right)n$. By Lemma 15 there are at least

$$\frac{1}{2}\left((s-1)\left(\frac{n-1}{t}-\sqrt{6(n-1)\frac{\log n}{t}}\right)+\frac{(n-s+1)(s-1)}{t}-(n-s+1)\sqrt{6(s-1)\frac{\log n}{t}}\right)$$

edges in G_i between the parts of π. If π has at most $s-2$ colors in G_i between the parts, then

$$(s-2)\left(\frac{n}{2t}+4\sqrt{\frac{n}{t}\log n}\right) \geq \frac{s-1}{2}\left(\frac{n-1}{t}-\sqrt{6(n-1)\frac{\log n}{t}}+\frac{(n-s+1)}{t}-(n-s+1)\sqrt{\frac{6\log n}{(s-1)t}}\right).$$

Rearranging we have

$$\frac{s-2}{s-1}\left(\frac{n}{t}+8\sqrt{\frac{n}{t}\log n}\right)+\frac{1}{t}+\sqrt{6(n-1)\frac{\log n}{t}}+(n-s+1)\sqrt{\frac{6\log n}{(s-1)t}} \geq \frac{n}{t}+\frac{(n-s+1)}{t}.$$

We will give an upper bound to the left side and a lower bound to the right side that give a contradiction.

Since s is an integer and $n/2+1 < s$, we have

$$(n-s+1)\sqrt{\frac{6}{n(s-1)}} \leq \frac{n}{2}\sqrt{\frac{12}{n^2}} = \sqrt{3}. \tag{†}$$

Therefore

$$\frac{s-2}{s-1}\left(\frac{n}{t}+8\sqrt{\frac{n}{t}\log n}\right)+\frac{1}{t}+\sqrt{6(n-1)\frac{\log n}{t}}+(n-s+1)\sqrt{\frac{6\log n}{(s-1)t}}$$
$$\leq \sqrt{C}\log n\left(\frac{n}{n-C\log n}\right)\left(\sqrt{C}+\frac{\sqrt{C}}{n}+\sqrt{\frac{n-C\log n}{n}}\left(8+\sqrt{\frac{6(n-1)}{n}}+(n-s+1)\sqrt{\frac{6}{n(s-1)}}\right)\right)$$

$$\leq \sqrt{C}\log n\left(\frac{n}{n-C\log n}\right)\left(\sqrt{C}+\frac{\sqrt{C}}{n}+\sqrt{\frac{n-C\log n}{n}}\left(8+\sqrt{6}+\sqrt{3}\right)\right) \quad \text{by } (\dagger).$$

Since $C = 1000$ and $n \geq 1,000,000$, $\frac{n}{n-C\log n} \leq 1.02$ and $\sqrt{\frac{n}{n-C\log n}} \leq 1.01$. Thus the term above is bounded by

$$\sqrt{C}\log n\left(1.02\sqrt{C}+\frac{1.02\sqrt{C}}{n}+1.01(8+\sqrt{6}+\sqrt{3})\right) \leq \sqrt{C}\log n\left(1.02\sqrt{C}+12.31\right).$$

We next bound the right side. By $(*)$ we have $\frac{1}{t} \geq \frac{C\log n}{n}$, and since $s \leq \left(1-\frac{14}{\sqrt{C}}\right)n$, so

$$\frac{n}{t}+\frac{(n-s+1)}{t} \geq C\log n + C\log n\frac{n-s+1}{n} \geq C\log n + C\log n\frac{14}{\sqrt{C}} = \sqrt{C}\log n(\sqrt{C}+14).$$

When $C = 1000$ we have $\sqrt{C}+14 > 1.02\sqrt{C}+12.31$, which gives a contradiction. So, there must be at least $s-1$ colors in G_i between the parts of π when $\frac{n}{2}+1 < s \leq \left(1-\frac{14}{\sqrt{C}}\right)n$.

Suppose $2 \leq s \leq \frac{n}{2}+1$. By Lemma 15 there are at least

$$\frac{1}{2}\left((s-1)\left(\frac{n-1}{t}-\sqrt{6(n-1)\frac{\log n}{t}}\right)+\frac{(n-s+1)(s-1)}{t}-(s-1)\sqrt{6(n-s+1)\frac{\log n}{t}}\right)$$

edges in G_i between the parts of π. If π has at most $s-2$ colors in G_i between the parts then

$$(s-2)\left(\frac{n}{2t}+4\sqrt{\frac{n}{t}\log n}\right) \geq \frac{(s-1)}{2}\left(\frac{n-1}{t}-\sqrt{6(n-1)\frac{\log n}{t}}+\frac{(n-s+1)}{t}-\sqrt{6(n-s+1)\frac{\log n}{t}}\right).$$

Rearranging we have

$$\frac{s-2}{s-1}\left(\frac{n}{t}+8\sqrt{\frac{n}{t}\log n}\right)+\frac{1}{t}+\sqrt{6(n-1)\frac{\log n}{t}}+\sqrt{6(n-s+1)\frac{\log n}{t}} \geq \frac{n}{t}+\frac{(n-s+1)}{t}.$$

Using $\frac{1}{t} \leq \frac{C\log n}{n-C\log n}$ from $(*)$, we have

$$\frac{s-2}{s-1}\left(\frac{n}{t}+8\sqrt{\frac{n}{t}\log n}\right)+\frac{1}{t}+\sqrt{6(n-1)\frac{\log n}{t}}+\sqrt{6(n-s+1)\frac{\log n}{t}}$$
$$\leq \sqrt{C}\log n\left(\frac{n}{n-C\log n}\right)\left(\sqrt{C}+\frac{\sqrt{C}}{n}+\sqrt{\frac{n-C\log n}{n}}\left(8+\sqrt{\frac{6(n-1)}{n}}+\sqrt{\frac{6(n-s+1)}{n}}\right)\right).$$

Since $C = 1000$ and $n \geq 1,000,000$, $\frac{n}{n-C\log n} \leq 1.02$ and $\sqrt{\frac{n}{n-C\log n}} \leq 1.01$. Thus the term above is bounded above by

$$\sqrt{C}\log n\left(1.02\sqrt{C}+\frac{1.02\sqrt{C}}{n}+1.01\left(8+2\sqrt{6}\right)\right) \leq \sqrt{C}\log n\left(1.02\sqrt{C}+13.1\right).$$

Bounding the right side using $\frac{1}{t} \geq \frac{C\log n}{n}$ from $(*)$, and $s \leq \frac{n}{2}+1$, we have

$$\frac{n}{t}+\frac{(n-s+1)}{t} \geq C\log n + C\log n\frac{(n-s+1)}{n} \geq C\log n + C\log n\frac{\frac{n}{2}}{n} = \sqrt{C}\log n\left(\frac{3\sqrt{C}}{2}\right).$$

Again, when $C = 1000$ and $n \geq 1,000,000$ we have $\frac{3\sqrt{C}}{2} > 1.02\sqrt{C}+13.1$ which leads to a contradiction. Thus, there must be at least $s-1$ colors in G_i between the parts of π when $2 \leq s \leq \frac{n}{2}+1$. □

The careful reader will note that, in the proof of Lemma 16, the value of C cannot be taken too *large*, as well as too small, and this seems counterintuitive - the larger the value of C is the smaller the number of spanning trees we ask for. The essential reason for this is simply the fact that, in the proof, we need to control $\frac{1}{t} = \frac{1}{\lfloor n/(C \log n) \rfloor}$ in comparison to $C \log n/n$ and this can run awry if $n/(C \log n)$ is too small. This leads to an interplay between C and n. Taking larger C is allowed within the scope of the proof so long as n is taken to be sufficiently large as well, but we have made some attempt to optimize so that C and n are relatively small.

6 Main Result

Theorem 1. *Let G be an edge-colored copy of K_n, where each color appears on at most $n/2$ edges and $n \geq 1,000,000$. The graph G contains at least $\lfloor n/(1000 \log n) \rfloor$ edge-disjoint rainbow spanning trees.*

Proof. Recall that $t = \lfloor n/(C \log n) \rfloor$ where $C = 1000$. We perform the random experiment of decomposing the edges of G into t edge-disjoint subgraphs G_i by independently and uniformly selecting each edge of G to be in the subgraph G_i with probability $1/t$. With probability at least $1 - \frac{7}{n}$ none of the bad events from Lemmas 6, 8, 9, and 10 occur in any of the subgraphs G_i. Henceforth let $G_1, \ldots, G_t$ be fixed subgraphs where none of these bad events occur.

We want to show that each G_i has a rainbow spanning tree. By Theorem 2 it is enough to show that for every partition π of $V(G)$ into s parts, there are at least $s - 1$ different colors appearing on the edges of G_i between the parts of π.

By Lemma 6, every G_i has at least one edge from each of the $n - 1$ pseudocolor classes. When $s = n$ there must be at least $n - 1$ colors in G_i between the parts of π. When $s = n - 1$ there is at most one edge inside the parts of π, so there are at least $n - 2$ colors in G_i between the parts of π.

If $\left(1 - \frac{14}{\sqrt{C}}\right) n \leq s \leq n - 2$, then by Lemma 8 every partition π of $V(G)$ into s parts has at least $s - 1$ colors in G_i between the parts, for every subgraph $G_1, \ldots, G_t$.

Finally, we assume that $s \leq \left(1 - \frac{14}{\sqrt{C}}\right) n$. When $s = 1$ there are zero colors between the parts, so the condition is vacuously true. So suppose $2 \leq s \leq \left(1 - \frac{14}{\sqrt{C}}\right) n$. Since Lemmas 9 and 10 hold, by Lemma 16 the number of colors between the parts of π is at least $s - 1$ for every subgraph $G_1, \ldots, G_t$.

Therefore all of the subgraphs $G_1, \ldots, G_t$ contain a rainbow spanning tree, and so G contains at least $t = \lfloor n/(1000 \log n) \rfloor$ edge-disjoint rainbow spanning trees. □

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