A VARIETY OF STEINER LOOPS SATISFYING MOUFANG’S THEOREM: A SOLUTION TO RAJAH’S PROBLEM

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Abstract. A loop $X$ is said to satisfy Moufang’s theorem if for every $x, y, z \in X$ such that $x(yz) = (xy)z$ the subloop generated by $x, y, z$ is a group. We prove that the variety $V$ of Steiner loops satisfying the identity $(xz)(((xy)z)(yz)) = ((xz)((xy)z))(yz)$ is not contained in the variety of Moufang loops, yet every loop in $V$ satisfies Moufang’s theorem. This solves a problem posed by Andrew Rajah.

1. Introduction

A Moufang loop is a loop satisfying the identity $x(y(xz)) = ((xy)x)z$. Moufang’s theorem [4] states that in every Moufang loop $X$ the implication

\[(1) \quad x(yz) = (xy)z \implies \langle x, y, z \rangle \text{ is a group}\]

holds for every $x, y, z \in X$. A short proof of Moufang’s theorem can be found in [3].

We say that a loop $X$ satisfies Moufang’s theorem if (1) holds for every $x, y, z \in X$, and a class $\mathcal{X}$ of loops satisfies Moufang’s theorem if every $X \in \mathcal{X}$ satisfies Moufang’s theorem.

In 2011, Andrew Rajah asked [7]: Is there a variety of loops not contained in the variety of Moufang loops that satisfies Moufang’s theorem? In this short note we give an affirmative answer to Rajah’s question.

Some classes of non-Moufang loops satisfying Moufang’s theorem are known. In [1], Colbourn et al observed that a Steiner loop satisfies Moufang’s theorem if and only if its corresponding Steiner triple system has the property that every Pasch configuration generates a subsystem of order 7, i.e., a Fano plane. They also determined the spectrum of finite orders for which there exist non-Moufang Steiner loops satisfying Moufang’s theorem. In [6], Stuhl proved that all oriented Hall loops of exponent 4 satisfy Moufang’s theorem. Our solution follows a similar line of reasoning. Marina Rasskazova recently announced an independent solution [5].

Recall that a Steiner loop [2] is a loop satisfying the identities

\[(2) \quad xy = yx \quad \text{and} \quad x(xy) = y.\]

A Steiner quasigroup is a quasigroup satisfying the identities

\[(3) \quad xy = yx, \quad x(xy) = y \quad \text{and} \quad xx = x.\]

There is a one-to-one correspondence between Steiner loops and Steiner quasigroups. Given a Steiner loop $X$ with identity element 1, the corresponding Steiner quasigroup is obtained by removing 1 and setting $xx = x$ for every $x \in X \setminus \{1\}$. Conversely, given a Steiner
quasigroup $X$, the corresponding Steiner loop is obtained by adjoining a new element 1 and setting $x1 = 1x = x$ for every $x \in X \cup \{1\}$.

A Steiner triple system is a partition of the edges of a complete graph into edge-disjoint triangles, with the vertices called points and the triangles called blocks. There is a one-to-one correspondence between Steiner triple systems and Steiner quasigroups. Given a Steiner triple system $S$ on $X$, the corresponding Steiner quasigroup $(X, \cdot)$ is defined by setting $xy = z$ if $\{x, y, z\}$ is a block of $S$ and $xx = x$ for every $x \in X$. Conversely, given a Steiner quasigroup $(X, \cdot)$, the corresponding Steiner triple system $STS(X)$ is obtained by declaring $\{x, y, z\}$ to be a block whenever $x \neq y$ satisfy $xy = z$.

Let $X$ be a Steiner quasigroup and $S = STS(X)$. Suppose that $x, y, z \in X$ are three points not contained in a block of $S$. Then $x(yz) = (xy)z$ holds if and only if the three points $x, y, z$ give rise to the Pasch configuration in Figure 1, with $c = x(yz) = (xy)z$. There is, of course, also the block $\{x, z, xz\}$, which is usually not depicted in a Pasch configuration. If, in addition, $x(yz) = y(xz)$ and $(xy)(yz) = xz$, then the three points $x, y, z$ give rise to the Fano plane in Figure 1. Note that both $x(yz) = y(xz)$ and $(xy)(yz) = xz$ can be interpreted as instances of associativity in Steiner quasigroups, namely, $(yz)x = x(yz) = y(xz) = y(xz)$ and $(xy)(yz) = xz = x(y(yz))$.

Consider now the corresponding Steiner loop on $X \cup \{1\}$. If $x, y \in X$ are such that $x \neq y$ then $\langle x, y \rangle$ is a Klein group. If $x, y, z \in X$ are three distinct points that give rise to a Fano plane then $\langle x, y, z \rangle$ is an elementary abelian 2-group of order 8.

2. Solution to Rajah’s problem

**Proposition 2.1.** A Steiner loop $X$ satisfies Moufang’s theorem if and only if

\[(4)\quad x(yz) = (xy)z \implies x(yz) = y(xz)\]

holds for every $x, y, z \in X$.

**Proof.** If $X$ is a Steiner loop satisfying Moufang’s theorem and $x(yz) = (xy)z$ for some $x, y, z \in X$, then $\langle x, y, z \rangle$ is a commutative group and hence $x(yz) = y(xz)$. Conversely, suppose that $X$ is a Steiner loop satisfying (4) and let $x, y, z \in X$ be such that

\[(5)\quad x(yz) = (xy)z.\]

If $1 \in \{x, y, z\}$ or $\{x, y, z\}$ is contained in a block then $\langle x, y, z \rangle$ is a group. For the rest of the proof suppose that $x, y, z$ are distinct non-identity elements not contained in a block so that
the three points $x$, $y$, $z$ form a Pasch configuration as in Figure 1. By (4), $x(yz) = y(xz)$ and $\{y, xz, x(yz)\}$ is a block. Furthermore, with $u = x$, $v = xy$ and $w = (xy)z$ we have
\[ u(vw) = xz = y(y(xz)) \overset{(4)}{=} y(x(yz)) = (x(xy))((xy)z) \overset{(5)}{=} (x(xy))((xy)(yz)) = (uv)w. \]
Applying (4) with $(u, v, w)$ in place of $(x, y, z)$, we obtain
\[ xz = u(vw) = v(uw) = (xy)(x((xy)z)) \overset{(5)}{=} (xy)(x(x(yz))) = (xy)(yz). \]
This shows that $\{xy, yz, xz\}$ is a block and thus $\langle x, y, z \rangle$ is a group. □

**Lemma 2.2.** Any Steiner loop satisfying the identity
\[ (xz)(((xy)z)(yz)) = ((xz)((xy)z))(yz) \]

satisfies Moufang’s theorem.

**Proof.** Suppose that $X$ is a Steiner loop satisfying (6). By Proposition 2.1, it suffices to check that (4) holds. Let $x, y, z \in X$ be such that (5) holds. Then
\[ z = (xz)x = (xz)(((xy)z)(yz)) \overset{(5)}{=} (xz)(((xy)z)(yz)) \]
\[ \overset{(6)}{=} ((xz)((xy)z))(yz) \overset{(5)}{=} (xz(x(yz)))(yz). \]
Multiplying by $yz$ then yields $y = (xz)(x(yz))$, multiplying further by $xz$ yields $(xz)y = x(yz)$, and we obtain $x(yz) = y(xz)$ by commutativity. □

There is a unique Steiner triple system of order 9, namely the affine triple system of order 9 with the corresponding Steiner quasigroup $(\mathbb{Z}_3 \times \mathbb{Z}_3, \cdot)$, $x \cdot y = -x - y$.

**Theorem 2.3.** Let $V$ be the variety of Steiner loops satisfying the identity (6). Then $V$ satisfies Moufang’s theorem and it is not contained in the variety of Moufang loops.

**Proof.** By Lemma 2.2, every $X \in V$ satisfies Moufang’s theorem. It can be checked that the Steiner loop of order 10 satisfies (6) but is not Moufang. □

Note that the Steiner loop of order 10 satisfies identities that are not consequences of (2) and (6), for instance the identity $(xy)(y(xz)) = x(y((xy)z))$.

**Problem 2.4.** Describe the equational theory of the variety of Steiner loops generated by the Steiner loop of order 10.

**References**


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