A PROBLEM IN FAIR DIVISION

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This problem was posed in the Winter 2004 Mathematics Alumni Newsletter:

A rectangular hole is cut from the interior of a rectangular pizza. What is the minimal number of cuts dividing the pizza into two equal shares?

Kari and I thought about it during supper. We found the solution in a few minutes. It turns out that the solution is easy once you see it. This is an account of our thought process leading to the solution. It is not meant to be of scientific value.

Since the illustration accompanying the problem looked as in Figure 1, we were influenced by it and came up with our first solution (cf. Figure 2). In these and all other illustrations, if a line appears to be parallel or perpendicular to another line, it is meant to be. If a point appears to be a midpoint of a line segment, it is meant to be, etc.

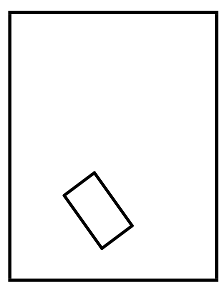

FIGURE 1. Rectangular pizza with a rectangular hole.

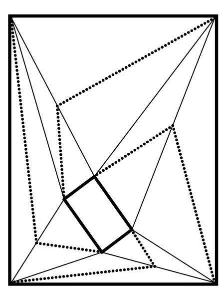

FIGURE 2. Splitting triangles.

Hence our first idea was to cut the pizza into triangular regions, and then split all triangular regions evenly. This only requires to be able to find a midpoint of a line segment. Note that we need a meager 16 cuts.

We were not happy splitting triangles and we came up with a solution that does not require it (Figure 3.) The four triangles don't have to be split since two and two of them are identical. We were happy with this solution for a while. Note that we only need 8 cuts.

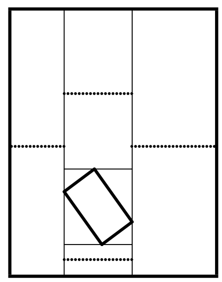

FIGURE 3. No splitting of triangles required.

Then I came up with a devilish solution (Figure 4). Cut the pizza in half. Place the half with the hole on top of the other half and use the hole as a template. Since the problem did not prohibit folding the pizza nor placing a part of the pizza on top of the rest, this seemed to be a clever solution. It requires 6 cuts.

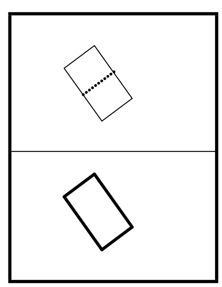

FIGURE 4. Placing pieces on top of each other.

Kari countered with a much better solution (Figure 5). Its first step is to construct a line through the center of the hole and one of the corners of the pizza. It requires only 3 cuts! (In Figure 5 it looks as if the first cut were perpendicular to the sides of the hole. It does not have to be.)

We realized at this point that it is clever to cut through the center of the hole, since any such cut will split the *hole* evenly. Hence it has something to do with splitting the *pizza* evenly. It was then easy to see that what we want is a cut through the center of the hole such that the two line segments on the horizontal outline of the pizza are divided alike (Figure 6).

But can it be done? It appears that in order to do this, we must be able to construct a line segment of length a. Surely, we can try to cut through the center

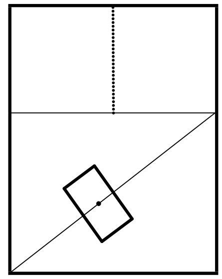

FIGURE 5. Cutting through the center of the hole and a corner of the pizza.

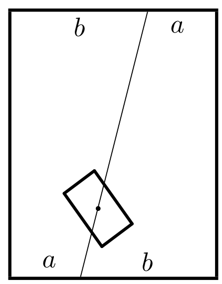

FIGURE 6. A single cut?

of the hole and see if the two line segments are of the same length. Thus we can get arbitrarily close to a fair division by a single cut.

But can we really divide the pizza equally by a single cut? What follows is a nice example of the danger of over-education. I reasoned as follows. It can so happen that the dimensions of the pizza and the hole are such that $a = \pi$. Thus, we are effectively constructing π. It is well-known (and proved in our course Introduction to Algebraic Structures I) that π cannot be constructed by compass and straightedge. (We are for the first time considering the tools allowed. More on this later.) Thus it appears to be hopeless. Perhaps the solution with three cuts is the best. I started to worry about 2-cut solutions.

But then it occurred to me that it is possible to construct π if you start with something that is related to π. For instance, if you are *given* a line segment of length 2π, you construct π by dividing that line segment equally. Isn't is so that if $a = \pi$ in Figure 6 then π is somehow encoded in the dimensions of the pizza or the hole?

This seemed hopelessly hard to solve. Then we finally realized that a single cut through a rectangle divides the rectangle evenly if and only if it passes through the center of the rectangle (can you prove it?).*Eureka*, all we need is to determine the center of the hole, the center of the pizza, and then construct the line passing through these two points. The fact that the horizontal outlines of the pizza are divided alike is then a consequence.

Are we done? Well, the only glitch is this: the center of the hole can coincide with the center of the pizza, in which case we do not have enough information to construct the line. Figure 1 does not look like that, but it is only an illustration in the end. However, if the two centers coincide, we can cut the pizza evenly by any line passing through the (coinciding) centers.

We can therefore remove the question mark from the caption of Figure 6: *A single cut suffices.*

Some questions

What do you think of this nice engineering solution? *Lift the pizza by a corner and let it hang down by its own weight. Then determine the line perpendicular to the ground that passes through the corner by which you are holding the pizza. This line is your fair cut.* Or is it? Does your answer change if you assume that the weight of the pizza is evenly distributed with respect to the surface area?

Finally, let us return to the question of the tools are allowed. Note that in our final solution we needed to find the center of a rectangle. This can be done by connecting the two midpoints of the two parallel sides of the rectangle, in both directions. Hence we only need to find a midpoint of a line segment (this can be done easily with compass) and draw a line through 2 points (done with straightedge). These are the classical tools of constructible Euclidean geometry.

Let me pose three questions that should be easy to answer but I don't see the answer immediately.

Let us say that a *single application of a compass* is this: use two known points to set up the radius r, then use a known point P as a center, then draw a circle centered at P of radius r. A *single application of a straightedge* is this: given two known points P, Q, construct an infinite line passing through these two points.

We can declare intersections of newly constructed objects as new known points and use them in subsequent steps. In this way, many geometrical problems can be solved by straightedge and compass. For instance, it is easy to show that one can:

(i) construct the midpoint of a given line segment,
(ii) construct the line perpendicular to a given line and passing through a given point,
(iii) construct the line parallel to a given line and passing through a given point.

My question is: what is the minimal number of applications of a straightedge and compass to achieve (i), (ii), (iii)? I can visualize solutions with 3, 4 and 3 applications, respectively. Can you do better? (I bet the answers are known. I did not do any search.)