The construction of loops by varying group tables

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The object of this talk is a discussion of the ways in which loops can be constructed from groups by using combinatorial ideas and symmetry. The starting point is the observation by the presenter and P. Vojtechovsky that the multiplication tables of groups with respect to right division are much more symmetrical than the ordinary tables, in that they can be written as block circulants with further symmetry properties. A weakening of the strong symmetry of these tables produces tables corresponding to loops. For example, there is a nice construction for the Moufang loop of order 12 (using the right division operation) as

1	3	2	4	5	6	7	8	9	10	11	12
2	1	3	6	4	5	9	7	8	12	10	11
3	2	1	5	6	4	8	9	7	11	12	10
4	6	5	1	2	3	10	12	11	7	9	8
5	4	6	3	1	2	12	11	10	9	8	7
6	5	4	2	3	1	11	10	12	8	7	9
7	9	8	10	12	11	1	2	3	4	6	5
8	7	9	12	11	10	3	1	2	6	5	4
9	8	7	11	10	12	2	3	1	5	4	6
10	12	11	7	9	8	4	6	5	1	2	3
11	10	12	9	8	7	6	5	4	3	1	2
12	11	10	8	7	9	5	4	6	2	3	1

This can be written more briefly as

$C(1,3,2)$	$C(4,5,6)$	$C(7,8,9)$	$C(10,11,12)$
$C(4,6,5)$	$C(1,2,3)$	$RC(10,12,11)$	$RC(7,9,8)$
$C(7,9,8)$	$RC(10,12,11)$	$C(1,2,3)$	$RC(4,6,5)$
$C(10,12,11)$	$RC(7,9,8)$	$RC(4,6,5)$	$C(1,2,3)$

where $C(i,j,k)$ denotes a circulant and $RC(i,j,k)$ a reverse circulant i.e. each row is obtained from the previous by a left shift. Diassociativity more or less fixes all the blocks except for those in the $(2,3)$, $(2,4)$ and $(3,4)$ positions, together with their reflections in the diagonal, and in these positions the reverse circulants appear. The loops from the Chein construction based on dihedral groups can be presented in an analogous way. It is an interesting question as to whether the Moufang condition may be verified by showing the closure of the corresponding diagrams in web geometry, which of course can be translated into the conditions on the tables.

Although it is unlikely that simple loops of an interesting nature may be produced by some variations of symmetry as above, (for example it does not seem possible to construct the simple Moufang loop of order 120 in a similar way to that above) this naive approach can produce loops which automatically satisfy certain conditions, for example having a composition series of a certain type. Moreover, by varying tables coming from groups we can produce tables which automatically correspond to non-associative loops and which have character tables which are specified. The following example illustrates. The table (under right multiplication) for any dihedral group of order 2n may be written

$$\begin{array}{cc} C(1,n,\ldots,2) & C(n+1,n+2,\ldots,2n) \\ C(n+1,2n,\ldots,n+2) & C(1,2,\ldots,n) \end{array}$$

If σ is any non-identity permutation on $\{2,\ldots,n\}$ the table

$$\begin{array}{cc} C(1,n,\ldots,2) & C(n+1,n+2,\ldots,2n) \\ C(n+1,2n,\ldots,n+2) & C(1,\sigma(2),\ldots,\sigma(n)) \end{array}$$

corresponds to a non-associative loop. Question: how many distinct isomorphism classes of loops can be produced in this way and what identities do they satisfy?

If time permits I will also indicate other symmetrical constructions which produce families of loops.