Reflection quasigroups: Algebraic models for symmetric spaces with midpoints

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A reflection quasigroup $(Q, \cdot)$ is an idempotent quasigroup for which each left translation is an involutive automorphism. We give a geometric interpretation to such structures by viewing $p \cdot x = y$ to mean that the point reflection through p carries x to y. (The operation of point reflection in a vector space, $p \cdot x := 2p - x$, $2 \neq 0$, is a basic example.) From this perspective the unique solution $x = b/a$ of $x \cdot a = b$ yields the unique "midpoint" between a and b; this midpoint plays a crucial role in the theory. These structures turn out to have characterizations in terms of, or even categorical equivalences with, a variety of other algebraic structures: uniquely 2-divisible twisted subgroups, transversal twisted subgroups of involutive groups, a special class of loops called B-loops, and uniquely 2-divisible gyrocommutative gyrogroups.

We consider reflection quasigroups arising from a general class of symmetric spaces called lineated symmetric spaces. Our primary interest is the case that these symmetric spaces are (differentiable) Banach manifolds, in which case they exhibit an interesting geometric structure, and particularly in the metric case, where it is assumed the symmetric space carries a convex metric, an invariant complete metric contracting the square root function. One major result is that the distance function between points evolving over time on two geodesics is a convex function. Primary examples arise from involutive Banach-Lie groups (G, σ) admitting a polar decomposition $G = P \cdot K$, where K is the subgroup fixed by σ and P is the associated symmetric space. We consider an appropriate notion of seminegative curvature for such symmetric spaces endowed with an invariant Finsler metric and prove that the corresponding length metric must be a convex metric. The preceding results provide a general framework for the interesting Finsler geometry of the space of positive Hermitian elements of a C^*-algebra that has emerged in recent years.