Right versus Left:
Relativistic dynamics in the Einstein velocity loop

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We explore the loop $(D, \oplus_E)$ of Einstein velocity addition on the ball $D = \{v \in \mathbb{R}^3 : |v| \leq c\}$ of relativistically admissible velocities. This loop, which is also a gyrocommutative gyrogroup, can be used to derive the relativistic dynamics equation that describes, for example, the evolution of the velocity of a charged particle in an electromagnetic field.

The natural first step in developing dynamics from the loop $(D, \oplus_E)$ is to look at the *translations* of the loop. However, since the addition $\oplus_E$ is, in general, not commutative (in fact, it is commutative only for parallel velocities), one must decide whether to use the *right* or the *left* translations. There seems to be no *a priori* preference for either one. Indeed, one can ask whether it makes a difference. Is the dynamics which stems from the left translations different from the dynamics of the right translations? If so, which dynamics does nature choose?

We will show how the *left* translations $\varphi_a : D \to D$, $\varphi_a(v) = a \oplus_E v$ lead to the usual relativistic dynamics equation. The development here is straightforward, due mainly to two facts:

- the left translations of $(D, \oplus_E)$ are *projective* automorphisms of D
- the inverse of the left translation φ_a is again a left translation, namely φ_{-a}.

In contrast, the *right* translations $\phi_a : D \to D$, $\phi_a(v) = v \oplus_E a$ are problematic. They are not projective maps. They're not even analytic! And the inverse of a right translation is not a right translation. A closer look at the *physical meaning* of right and left translations reveals that there is an inherent difference between them. We will explain this asymmetry and discuss possible directions for developing the "right" dynamics.