Triality

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The Main Actor

symmetric group \( S_3 \)
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generators: $\sigma = (1\ 2)$ $\rho = (1\ 2\ 3)$
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The Main Actor

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$\sigma^2 = 1$, \quad $\rho^3 = 1$, \quad $\sigma\rho\sigma = \rho^{-1} = \rho^2$
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triality: There is an $S_3$-action and ...
Lie algebras with triality

(Mikheev '92) \( g \) is a Lie algebra with triality if
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- \( \text{Aut}(g) \supseteq S = \langle \sigma, \rho \rangle \cong S_3 \)
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Equivalently:

$$\begin{align*}
(\ast) \quad (x^\sigma - x) + (x^\sigma - x)^\rho + (x^\sigma - x)^{\rho^2} &= 0 \\
\forall \ x \in \mathfrak{g}
\end{align*}$$
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Ex: $\mathfrak{g} = \mathfrak{o}(\emptyset, n)$ type D$_4$
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**Ex:** \( \mathfrak{g} = \mathfrak{o}(\mathbb{O}, n) \) type D\(_4\)

\( \mathfrak{o}(\mathbb{O}, n) = \text{Der}(\mathbb{O}) \oplus \{ L_a \mid a \in \mathbb{O}_0 \} \oplus \{ R_a \mid a \in \mathbb{O}_0 \} \).
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\[
\begin{align*}
  d^\sigma &= d, & L_a^\sigma &= -R_a, & R_a^\sigma &= -L_a \\
  d^{\rho} &= d, & L_a^{\rho} &= R_a, & R_a^{\rho} &= -L_a - R_a
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(Mikheev '92) \( \mathfrak{g} \) is a Lie algebra with \textit{triality} if

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\]

Then \( L_a^\sigma - L_a = -R_a - L_a, \quad (L_a^\sigma - L_a)^\rho = L_a, \quad (L_a^\sigma - L_a)^{\rho^2} = R_a \)
so sum = 0.
Groups with triality

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Results on groups with triality via connections with loops:
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Results on groups with triality via connections with loops:

- Glauberman ('68)
- Doro ('78)
- Grishkov-Zavarnitsine ('06)
- Hall ('10)
- B,M,P-I ('13)
Moufang Loops (Moufang ’35)

**loop:** A set $M$ with a product $(a, b) \mapsto ab$ such that
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Groups with Triality and Moufang Loops

- $G$ a group with triality $\implies$
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- $G$ a group with triality $\implies$

$$\mathcal{M}(G) := \{g^{-1}g^{\sigma} \mid g \in G\}$$

is a Moufang loop w.r.t.
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$m \cdot n = m^{-\rho} nm^{-\rho^2} = n^{-\rho^2} mn^{-\rho} \ \forall \ m, n \in \mathcal{M}(G)$
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$$L_m \xrightarrow{\rho} R_m \xrightarrow{\rho} P_m \xrightarrow{\rho} L_m$$
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Groups with Triality and Moufang Loops

• \( G \) a group with triality \( \Rightarrow \)

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\[ L_m \xrightarrow{\rho} R_m \xrightarrow{\rho} P_m \xrightarrow{\rho} L_m \]

\[ P_m^\sigma = P_m^{-1}, \quad L_m^\sigma = R_m^{-1}, \quad R_m^\sigma = L_m^{-1} \]

J. Hall, *Moufang Loops and Groups with Triality are Essentially the Same Thing*
Lie algebras with triality

Groups with triality $\xrightarrow{\mathcal{M}(G)}$ Moufang loops $\xleftarrow{\mathcal{D}(M)}$
A Malcev algebra is a vector space $m$ with
A Malcev algebra is a vector space $m$ with a bilinear map $[\cdot, \cdot] : m \times m \rightarrow m$ s.t.
• A Malcev algebra is a vector space $m$ with a bilinear map $[\cdot, \cdot] : m \times m \to m$ s.t.

- $[x, y] = -[y, x]$,
- $[J(x, y, z), x] = J(x, y, [x, z])$
A Malcev algebra is a vector space $m$ with a bilinear map $[\cdot, \cdot] : m \times m \to m$ s.t.

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where $J(x, y, z) = [[x, y], z] + [[y, z], x] + [[z, x], y]$
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Examples:
Malcev Algebras (Malcev ’55)

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- Examples:
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where $J(x, y, z) = [[x, y], z] + [[y, z], x] + [[z, x], y]$

• Examples:

  (1) any Lie algebra

  (2) any alternative algebra under $[x, y] = xy - yx$
A Malcev algebra is a vector space \( m \) with a bilinear map \([\cdot, \cdot] : m \times m \to m\) s.t.

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[x, y] &= -[y, x], \\
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\end{align*}
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where \( J(x, y, z) = [[[x, y], z] + [[y, z], x] + [[z, x], y] \)

**Examples:**

1. any Lie algebra
2. any alternative algebra under \([x, y] = xy - yx\)
3. imaginary octonions under \([x, y] = xy - yx\)
\( m: \) a Malcev algebra of characteristic \( \neq 2,3 \)
m: a Malcev algebra of characteristic $\neq 2,3$

(Pérez-Izquierdo, Shestakov ’04)
\( m: \) a Malcev algebra of characteristic \( \neq 2, 3 \)

(Pérez-Izquierdo, Shestakov '04)

\( \mathcal{L}(m): \) Lie algebra generated by symbols \( \ell_x, r_x, \) for \( x \in m \) s.t. \( x \mapsto \ell_x, \ x \mapsto r_x \) are bilinear and
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\begin{align*}
(\text{a}) & \quad [\ell_x, \ell_y] = \ell_{[x,y]} - 2[\ell_x, r_y], \\
(\text{b}) & \quad [r_x, r_y] = -r_{[x,y]} - 2[\ell_x, r_y], \\
(\text{c}) & \quad [\ell_x, r_y] = [r_x, \ell_y].
\end{align*}
\]
Lie Algebras with Triality and Malcev Algebras

\( m \): a Malcev algebra of characteristic \( \neq 2, 3 \)

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(a) \quad [\ell_x, \ell_y] &= \ell_{[x,y]} - 2[\ell_x, r_y], \\
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**Thm.** \( \mathcal{L}(m) \) is a Lie algebra with triality w.r.t

- \( \ell^\sigma_x = -r_x, \quad r^\sigma_x = -\ell_x \)
- \( \ell^\rho_x = r_x, \quad r^\rho_x = -\ell_x - r_x \).
$m$: a Malcev algebra of characteristic $\neq 2, 3$

(Pérez-Izquierdo, Shestakov '04)

$L(m)$: Lie algebra generated by symbols $\ell_x, r_x$, for $x \in m$ s.t. $x \mapsto \ell_x$, $x \mapsto r_x$ are bilinear and

(a) $[\ell_x, \ell_y] = \ell_{[x,y]} - 2[\ell_x, r_y]$,
(b) $[r_x, r_y] = -r_{[x,y]} - 2[\ell_x, r_y]$,
(c) $[\ell_x, r_y] = [r_x, \ell_y]$.

**Thm.** $L(m)$ is a Lie algebra with triality w.r.t

- $\ell_x^\sigma = -r_x$, $r_x^\sigma = -\ell_x$
- $\ell_x^\rho = r_x$, $r_x^\rho = -\ell_x - r_x$. 
The Map II

Lie algebras with triality $\mathcal{L}(m) \leftrightarrow m$ Malcev algebras

Groups with triality $G \mapsto \mathcal{M}(G)$ Moufang loops $\mathcal{D}(M) \leftrightarrow M$
Cocommutative Hopf Algebras

► \((H, \Delta, \epsilon, S)\): a unital (cocommutative) Hopf algebra
Cocommutative Hopf Algebras

- \((H, \Delta, \epsilon, S)\): a unital (cocommutative) Hopf algebra

- \(\Delta(u) = \sum u_{(1)} \otimes u_{(2)} = \sum u_{(2)} \otimes u_{(1)} \quad \forall \ u \in H\)
(H, Δ, ε, S): a unital (cocommutative) Hopf algebra

Δ(u) = \sum u_{(1)} \otimes u_{(2)} = \sum u_{(2)} \otimes u_{(1)} \quad \forall \ u \in H

(Sweedler notation)
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Ex: \( U = U(\mathfrak{g}) \) (universal enveloping algebra of \( \mathfrak{g} \)): 
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  \((\text{Sweedler notation})\)

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- Ex: \(U = U(g)\) (universal enveloping algebra of \(g\)):

  \((a)\) \(\Delta : U \rightarrow U \otimes U, \quad \Delta(x) = x \otimes 1 + 1 \otimes x\)
Cocommutative Hopf Algebras

- \((H, \Delta, \epsilon, S)\): a unital (cocommutative) Hopf algebra

- \(\Delta(u) = \sum u_{(1)} \otimes u_{(2)} = \sum u_{(2)} \otimes u_{(1)} \quad \forall \ u \in H\)
  
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- Ex: \(U = U(\mathfrak{g})\) (universal enveloping algebra of \(\mathfrak{g}\)):

  \((a)\) \(\Delta : U \rightarrow U \otimes U, \quad \Delta(x) = x \otimes 1 + 1 \otimes x\)

  \((b)\) \(\epsilon : U \rightarrow \mathbb{F}, \quad \epsilon(x) = 0,\)
Cocommutative Hopf Algebras

- $(H, \Delta, \epsilon, S)$: a unital (cocommutative) Hopf algebra

- $\Delta(u) = \sum u(1) \otimes u(2) = \sum u(2) \otimes u(1) \quad \forall \ u \in H$  
  (Sweedler notation)

- $(\Delta \otimes \text{id})\Delta(u) = (\text{id} \otimes \Delta)\Delta(u) = \sum u(1) \otimes u(2) \otimes u(3)$

- **Ex:** $U = U(\mathfrak{g})$ (universal enveloping algebra of $\mathfrak{g}$):
  
  (a) $\Delta : U \rightarrow U \otimes U, \quad \Delta(x) = x \otimes 1 + 1 \otimes x$

  (b) $\epsilon : U \rightarrow \mathbb{F}, \quad \epsilon(x) = 0,$

  (c) $S : U \rightarrow U, \quad S(x) = -x,$

  for all $x \in \mathfrak{g}$
Defn. (B,M,P-I) A cocommutative Hopf algebra $H$ is a
Defn. (B,M,P-I) A cocommutative Hopf algebra $H$ is a Hopf algebra with triality if there exist $S_3 \cong \langle \sigma, \rho \rangle \subseteq \text{Aut}(H)$ s.t.
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$$\sum T(u_1) T(u_2)^\rho T(u_3)^{\rho^2} = \epsilon(u)1$$
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$$
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where $T(v) = \sum v_{(1)}^\sigma S(v_{(2)})$. 
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Ex. G: a group with triality w.r.t. $\sigma, \rho \in \text{Aut}(G)$
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$\mathbb{F}G$: group algebra with
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Ex. G: a group with triality w.r.t. $\sigma, \rho \in \text{Aut}(G)$

$\mathbb{F}G$ : group algebra with

$$\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1}$$

$\mathbb{F}G$ is a Hopf algebra with triality with $T(g) = g^\sigma g^{-1}$.

* Replace $T$ with $T'(g) = T(\sigma(g^{-1})) = g^{-1}g^\sigma$ to get earlier defn.
**Thm.** \( \mathfrak{g} \) is a Lie algebra with triality w.r.t. \( \sigma, \rho \) \( \implies \)
\( U(\mathfrak{g}) \) is a Hopf algebra with triality w.r.t. \( \sigma, \rho \).

**Proof.** \( T(\nu) = \sum \nu_{(1)}^\sigma S(\nu_{(2)}) \) where \( \Delta(\nu) = \sum \nu_{(1)} \otimes \nu_{(2)} \).
The Lie and Hopf Connection

**Thm.** \( \mathfrak{g} \) is a Lie algebra with triality w.r.t. \( \sigma, \rho \) \( \implies \)
\( \mathcal{U}(\mathfrak{g}) \) is a Hopf algebra with triality w.r.t. \( \sigma, \rho \).

**Proof.** \( T(v) = \sum v_1^{\sigma} S(v_2) \) where \( \Delta(v) = \sum v_1 \otimes v_2 \).

Now \( \Delta(x) = x \otimes 1 + 1 \otimes x, \) \( \epsilon(x) = 0, S(x) = -x \) for all \( x \in \mathfrak{g}, \)
and \( \Delta(1) = 1 \otimes 1, \) \( \epsilon(1) = 1, S(1) = 1. \)
Thm. \( g \) is a Lie algebra with triality w.r.t. \( \sigma, \rho \) \( \implies \) 
\( U(g) \) is a Hopf algebra with triality w.r.t. \( \sigma, \rho \).

Proof. \( T(v) = \sum v^\sigma_{(1)} S(v_{(2)}) \) where \( \Delta(v) = \sum v_{(1)} \otimes v_{(2)} \).

Now \( \Delta(x) = x \otimes 1 + 1 \otimes x, \epsilon(x) = 0, S(x) = -x \) for all \( x \in g \), and \( \Delta(1) = 1 \otimes 1, \epsilon(1) = 1, S(1) = 1 \).

\[
T(x) = x^\sigma S(1) + 1^\sigma S(x) = x^\sigma - x
\]
\[
T(1) = 1.
\]
The Lie and Hopf Connection

**Thm.** $\mathfrak{g}$ is a Lie algebra with triality w.r.t. $\sigma, \rho \implies U(\mathfrak{g})$ is a Hopf algebra with triality w.r.t. $\sigma, \rho$.

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Relation we want is $\sum T(u_{(1)}) T(u_{(2)})^{\rho} T(u_{(3)})^{\rho^2} = \epsilon(u)1$. 
The Lie and Hopf Connection

**Thm.** $\mathfrak{g}$ is a Lie algebra with triality w.r.t. $\sigma, \rho \implies U(\mathfrak{g})$ is a Hopf algebra with triality w.r.t. $\sigma, \rho$.

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Relation we want is $\sum T(u_{(1)}) T(u_{(2)})^\rho T(u_{(3)})^{\rho^2} = \epsilon(u)1$. Now

$$(\Delta \otimes \text{id})(\Delta(x)) = (\Delta \otimes \text{id})(x \otimes 1 + 1 \otimes x)$$

$$= x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x$$
The Lie and Hopf Connection

**Thm.** $g$ is a Lie algebra with triality w.r.t. $\sigma, \rho \implies U(g)$ is a Hopf algebra with triality w.r.t. $\sigma, \rho$.

**Proof.** $T(v) = \sum v_{(1)}^\sigma S(v_{(2)})$ where $\Delta(v) = \sum v_{(1)} \otimes v_{(2)}$.

Now $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\epsilon(x) = 0$, $S(x) = -x$ for all $x \in g$, and $\Delta(1) = 1 \otimes 1$, $\epsilon(1) = 1$, $S(1) = 1$.

$$T(x) = x^\sigma S(1) + 1^\sigma S(x) = x^\sigma - x$$

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Relation we want is $\sum T(u_{(1)}) T(u_{(2)})^\rho T(u_{(3)})^{\rho^2} = \epsilon(u)1$. Now

$$\left(\Delta \otimes \text{id}\right)(\Delta(x)) = \left(\Delta \otimes \text{id}\right)(x \otimes 1 + 1 \otimes x)$$

$$= x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x$$

$$(x^\sigma - x) + (x^\sigma - x)^\rho + (x^\sigma - x)^{\rho^2} = 0 = \epsilon(x)1. \quad \Box$$
Thm. H a Hopf algebra with triality w.r.t. \( \sigma, \rho \implies \)
Getting Some Arrows

Thm. H a Hopf algebra with triality w.r.t. $\sigma, \rho \Rightarrow$

(a) the primitive elements of $H$ form a Lie algebra with triality w.r.t. $\sigma, \rho$. 
Theorem H. A Hopf algebra with triality w.r.t. $\sigma, \rho$ implies:

(a) the primitive elements of $H$ form a Lie algebra with triality w.r.t. $\sigma, \rho$.

(Recall $y$ is primitive if $\Delta(y) = y \otimes 1 + 1 \otimes y$.)
Thm. \( H \) a Hopf algebra with triality w.r.t. \( \sigma, \rho \) \( \implies \)

(a) the primitive elements of \( H \) form a Lie algebra with triality w.r.t. \( \sigma, \rho \).

(Recall \( y \) is primitive if \( \Delta(y) = y \otimes 1 + 1 \otimes y \).)

(b) the group-like elements of \( H \) form a group with triality w.r.t. \( \sigma, \rho \).

(Recall \( g \) is group-like if \( \Delta(g) = g \otimes g \).)
Thm. \( H \) a Hopf algebra with triality w.r.t. \( \sigma, \rho \)

(a) the primitive elements of \( H \) form a Lie algebra with triality w.r.t. \( \sigma, \rho \).

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(b) the group-like elements of \( H \) form a group with triality w.r.t. \( \sigma, \rho \).

(Recall \( g \) is group-like if \( \Delta(g) = g \otimes g \).)
The Map III

Lie algebras with triality \[ \mathcal{L}(m) \leftrightarrow m \]

Malcev algebras

Hopf algebras with triality

\[ g \mapsto U(g) \]

primitive elts.

Groups with triality

\[ G \mapsto \mathcal{M}(G) \]

Moufang loops

\[ D(M) \leftrightarrow M \]
Moufang-Hopf Algebras

- $(U, \Delta, \epsilon)$: a (cocommutative) coassociative unital bialgebra s.t. for all $u, v, w \in U,$
(U, Δ, ε): a (cocommutative) coassociative unital bialgebra s.t. for all $u, v, w \in U$,

$$\sum u_{(1)} (v(u_{(2)} w)) = \sum ((u_{(1)} v)u_{(2)}) w$$
Moufang-Hopf Algebras

\[ (U, \Delta, \epsilon) : \text{ a (cocommutative) coassociative unital bialgebra} \]
s.t. for all \( u, v, w \in U \),

\[
\sum u_{(1)} (v (u_{(2)} w)) = \sum ((u_{(1)} v) u_{(2)}) w
\]

and there exists a linear map \( S : U \rightarrow U \) s.t.
(U, Δ, ε): a (cocommutative) coassociative unital bialgebra s.t. for all \( u, v, w \in U \),

\[
\sum u_{(1)} (v(u_{(2)}w)) = \sum ((u_{(1)}v)u_{(2)})w
\]

and there exists a linear map \( S: U \rightarrow U \) s.t.

\[
\sum S(u_{(1)}) (u_{(2)}v) = \varepsilon(u)v = \sum u_{(1)} (S(u_{(2)})v)
\]

\[
\sum (vu_{(1)}) S(u_{(2)}) = \varepsilon(u)v = \sum (vS(u_{(1)})) u_{(2)}.
\]
Moufang-Hopf Algebras

- $(U, \Delta, \epsilon)$: a (cocommutative) coassociative unital bialgebra s.t. for all $u, v, w \in U$, 

$$
\sum u_{(1)} (v (u_{(2)} w)) = \sum ((u_{(1)} v) u_{(2)}) w
$$

and there exists a linear map $S : U \rightarrow U$ s.t.

$$
\sum S(u_{(1)}) (u_{(2)} v) = \epsilon(u) v = \sum u_{(1)} (S(u_{(2)}) v)
$$

$$
\sum (vu_{(1)}) S(u_{(2)}) = \epsilon(u) v = \sum (vS(u_{(1)})) u_{(2)}.
$$

- In this case, say $(U, \Delta, \epsilon)$ is a Moufang-Hopf algebra.
Hopf Algebras With Triality & Moufang Hopf Algebras, the Map $U \mapsto \mathcal{O}(U)$

$U$: cocommutative Moufang-Hopf algebra
Hopf Algebras With Triality & Moufang Hopf Algebras, the Map $U \mapsto \mathfrak{D}(U)$

$U$: cocommutative Moufang-Hopf algebra

$\mathfrak{D}(U)$: unital associative algebra generated by $\{L_u, R_u, P_u \mid u \in U\}$
U: cocommutative Moufang-Hopf algebra

\( D(U) \): unital associative algebra generated by \( \{L_u, R_u, P_u \mid u \in U\} \)
where \( P_1 = 1 \), \( P_{\alpha u + \beta v} = \alpha P_u + \beta P_v \) and
Hopf Algebras With Triality & Moufang Hopf Algebras, the Map \(U \mapsto \mathfrak{D}(U)\)

\(U\): cocommutative Moufang-Hopf algebra

\(\mathfrak{D}(U)\): unital associative algebra generated by \(\{L_u, R_u, P_u \mid u \in U\}\)

where \(P_1 = 1\), \(P_{\alpha u + \beta v} = \alpha P_u + \beta P_v\) and

\[
\sum P_{u(1)} L_{u(2)} R_{u(3)} = \epsilon(u)1, \quad \sum R_{u(1)} P_v L_{u(2)} = P S(u)v
\]

\[
\sum L_{u(1)} P_v R_{u(2)} = P v S(u), \quad \sum P_{u(1)} P_v P_{u(2)} = \sum P_{u(1)} v u(2)
\]

& cyclic permutations of them \(P_u \xrightarrow{\rho} L_u \xrightarrow{\rho} R_u \xrightarrow{\rho} P_u\)
Hopf Algebras With Triality & Moufang Hopf Algebras, the Map $U \mapsto \mathcal{D}(U)$

$U$: cocommutative Moufang-Hopf algebra

$\mathcal{D}(U)$: unital associative algebra generated by $\{L_u, R_u, P_u \mid u \in U\}$

where $P_1 = 1$, $P_{\alpha u + \beta v} = \alpha P_u + \beta P_v$ and

\[
\sum P_{u_1} L_{u_2} R_{u_3} = \epsilon(u)1, \quad \sum R_{u_1} P_v L_{u_2} = P S(u) v
\]

\[
\sum L_{u_1} P_v R_{u_2} = P v S(u), \quad \sum P_{u_1} P_v P_{u_2} = \sum P_{u_1} v u_{(2)}
\]

& cyclic permutations of them $P_u \xrightarrow{\rho} L_u \xrightarrow{\rho} R_u \xrightarrow{\rho} P_u$

**Thm.** (B,M,P-I) $\mathcal{D}(U)$ with $\rho$ (above) and $\sigma$ given by $P_u \xrightarrow{\sigma} P S(u)$, $L_u \xrightarrow{\sigma} R S(u)$, $R_u \xrightarrow{\sigma} L S(u)$ is a Hopf algebra with triality.
The Map IV

Lie algebras with triality

$\mathfrak{g} \mapsto U(\mathfrak{g})$  primitive elts.

Hopf algebras with triality

$H \mapsto \mathcal{M}(H)$

Moufang-Hopf algebras

$G \mapsto \mathcal{M}(G)$

Moufang loops

Groups with triality

$\mathcal{L}(m) \leftrightarrow m$

Malcev algebras

$\mathcal{D}(U) \leftrightarrow U$

group algebra

$\mathcal{D}(M) \leftrightarrow M$

group-likes
Thm. (B,M,P-I) Let $H$ be cocommutative Hopf algebra with triality w.r.t. $\sigma, \rho$. Set
Thm. (B,M,P-I) Let $H$ be cocommutative Hopf algebra with triality w.r.t. $\sigma, \rho$. Set

$$T(x) = \sum x_{(1)}^\sigma S(x_{(2)})$$

for $x \in H$. 
Thm. (B,M,P-I) Let $H$ be cocommutative Hopf algebra with triality w.r.t. $\sigma, \rho$. Set

$$T(x) = \sum x_{(1)}^\sigma S(x_{(2)}) \quad \text{for } x \in H.$$ 

Then $\mathcal{M}(H) = \{ T(x) \mid x \in H \}$ is a Moufang-Hopf algebra with the coproduct, counit, and antipode inherited from $H$ and with product:
**Thm. (B,M,P-I)** Let $H$ be cocommutative Hopf algebra with triality w.r.t. $\sigma, \rho$. Set

$$T(x) = \sum x_1^\sigma S(x_2) \quad \text{for } x \in H.$$ 

Then $\mathcal{M}(H) = \{ T(x) \mid x \in H \}$ is a Moufang-Hopf algebra with the coproduct, counit, and antipode inherited from $H$ and with product:

$$u \ast v = \sum S(u_1)^\rho v S(u_2)^\rho = \sum S(v_1)^\rho u S(v_2)^\rho$$

for all $u, v \in \mathcal{M}(H)$. 

The Map V

Lie algebras with triality \( \mathcal{L}(m) \leftrightarrow m \)

Malcev algebras

Hopf algebras with triality

primitive elts. \( g \mapsto U(g) \)

Malcev-Hopf algebras

primitive elts. \( m \mapsto U(m) \)

Moufang-Hopf algebras

group algebra

Moufang loops

group-likes

loop algebra

D(U) \leftarrow U

D(M) \leftarrow M

G \mapsto \mathcal{M}(G)

H \mapsto \mathcal{M}(H)
Thm. (B,M,P-I) For any U cocommutative Moufang-Hopf algebra,
Thm. (B,M,P-I) For any $U$ cocommutative Moufang-Hopf algebra,

$\iota : U \rightarrow \mathcal{M}(\mathcal{D}(U)), \quad (\iota(u) = P_u)$
Thm. (B,M,P-I) For any U cocommutative Moufang-Hopf algebra,

\[ \iota : U \to M(D(U)), \quad \iota(u) = P_u \]

is an isomorphism of Moufang-Hopf algebras.
Thm. (B,M,P-I) For any U cocommutative Moufang-Hopf algebra,

\[ \iota : U \to \mathcal{M}(\mathcal{D}(U)), \quad (\iota(u) = P_u) \]

is an isomorphism of Moufang-Hopf algebras.

Thm. (B,M,P-I) Let \( \mathfrak{m} \) be a Malcev algebra of char. \( \neq 2, 3 \) (so \( \mathcal{U}(\mathfrak{m}) \) is a cocommutative Moufang-Hopf algebra). Then
Thm. (B,M,P-I) For any $U$ cocommutative Moufang-Hopf algebra,

\[ \iota : U \to \mathcal{M}(\mathcal{O}(U)), \quad (\iota(u) = P_u) \]

is an isomorphism of Moufang-Hopf algebras.

Thm. (B,M,P-I) Let $m$ be a Malcev algebra of char. $\neq 2, 3$ (so $\mathcal{U}(m)$ is a cocommutative Moufang-Hopf algebra). Then

\[ \mathcal{O}(\mathcal{U}(m)) \cong \mathcal{U}(\mathcal{L}(m)) \]
Thm. (B,M,P-I) For any $U$ cocommutative Moufang-Hopf algebra,

$$i : U \to M(D(U)), \quad (i(u) = Pu)$$

is an isomorphism of Moufang-Hopf algebras.

Thm. (B,M,P-I) Let $m$ be a Malcev algebra of char. $\neq 2, 3$ (so $U(m)$ is a cocommutative Moufang-Hopf algebra). Then

$$D(U(m)) \cong U(L(m))$$

$$U(m) \cong M(U(L(m))).$$
Connections with Universal Enveloping Algebras

Thm. (B,M,P-I) For any U cocommutative Moufang-Hopf algebra,

\[ \iota : U \to \mathcal{M}(\mathcal{D}(U)), \quad (\iota(u) = P_u) \]

is an isomorphism of Moufang-Hopf algebras.

Thm. (B,M,P-I) Let \( m \) be a Malcev algebra of char. \( \neq 2, 3 \) (so \( U(m) \) is a cocommutative Moufang-Hopf algebra). Then

\[ \mathcal{D}(U(m)) \cong U(L(m)) \]
\[ U(m) \cong \mathcal{M}(U(L(m))). \]

Remark. (Shestakov–Pérez-Izquierdo ('04))

\( m \) is a Lie algebra \( \implies U(m) = U(m). \)
Nichols Algebra \( N \)

generators: \( g, x_i \ (i = 1, \ldots, n) \)
Nichols Algebra N

- **generators:** \( g, x_i \ (i = 1, \ldots, n) \)
- **relations:** \( g^2 = 1, \ x_i x_j = -x_j x_i, \ gx_i = x_i g \)
Nichols Algebra $N$

generators: $g, x_i \ (i = 1, \ldots, n)$

relations: $g^2 = 1, \ x_i x_j = -x_j x_i, \ g x_i = x_i g$

$\Delta(g) = g \otimes g, \ \epsilon(g) = 1, \ S(g) = g^{-1} = g$
Nichols Algebra N

generators: \( g, x_i \ (i = 1, \ldots, n) \)

relations: \( g^2 = 1, \quad x_i x_j = -x_j x_i, \quad g x_i = x_i g \)

\[
\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1} = g
\]

\[
\Delta(x_i) = x_i \otimes g + 1 \otimes x_i, \quad \epsilon(x_i) = 0, \quad S(x_i) = -x_i g
\]
Nichols Algebra $N$

**generators:** $g, x_i \; (i = 1, \ldots, n)$

**relations:** $g^2 = 1, \quad x_i x_j = -x_j x_i, \quad g x_i = x_i g$

\[
\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1} = g
\]

\[
\Delta(x_i) = x_i \otimes g + 1 \otimes x_i, \quad \epsilon(x_i) = 0, \quad S(x_i) = -x_i g
\]

$\text{Aut}(N) = \text{GL}_n(F)$, \quad $a = (a_{i,j}) \in \text{GL}_n(F)$ \quad where
Nichols Algebra  N

generators:  \( g, x_i \ (i = 1, \ldots, n) \)

relations:  \( g^2 = 1, \ x_i x_j = -x_j x_i, \ g x_i = x_i g \)

\[
\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1} = g
\]

\[
\Delta(x_i) = x_i \otimes g + 1 \otimes x_i, \quad \epsilon(x_i) = 0, \quad S(x_i) = -x_i g
\]

\( \text{Aut}(N) = \text{GL}_n(\mathbb{F}), \quad a = (a_{i,j}) \in \text{GL}_n(\mathbb{F}) \) where

\[
\phi_a(g) = g, \quad \phi_a(x_i) = \sum_j a_{i,j} x_j
\]
Nichols Algebra \( \mathbb{N} \)

**generators:** \( g, x_i \ (i = 1, \ldots, n) \)

**relations:**

\[
g^2 = 1, \quad x_i x_j = -x_j x_i, \quad g x_i = x_i g
\]

\[
\begin{align*}
\Delta(g) &= g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1} = g \\
\Delta(x_i) &= x_i \otimes g + 1 \otimes x_i, \quad \epsilon(x_i) = 0, \quad S(x_i) = -x_i g
\end{align*}
\]

\( \text{Aut}(\mathbb{N}) = \text{GL}_n(\mathbb{F}) \), \( a = (a_{i,j}) \in \text{GL}_n(\mathbb{F}) \) where

\[
\phi_a(g) = g, \quad \phi_a(x_i) = \sum_j a_{i,j} x_j
\]

\( \sigma, \rho \) block diagonal w.r.t. \( \{ x_i | i = 1, \ldots, n \} \) with blocks
generators: \( g, x_i \ (i = 1, \ldots, n) \)

relations: \( g^2 = 1, \ x_ix_j = -x_jx_i, \ gx_i = x_ig \)

\[
\begin{align*}
\Delta(g) & = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1} = g \\
\Delta(x_i) & = x_i \otimes g + 1 \otimes x_i, \quad \epsilon(x_i) = 0, \quad S(x_i) = -x_ig 
\end{align*}
\]

\( \text{Aut}(N) = \text{GL}_n(\mathbb{F}) \), \( a = (a_{i,j}) \in \text{GL}_n(\mathbb{F}) \) where

\[
\phi_a(g) = g, \quad \phi_a(x_i) = \sum_j a_{i,j}x_j
\]

\( \sigma, \rho \) block diagonal w.r.t. \( \{x_i \mid i = 1, \ldots, n\} \) with blocks

\[
\begin{align*}
\sigma & : (1) \quad (-1) \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\rho & : (1) \quad (1) \quad \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \quad \omega^3 = 1.
\end{align*}
\]
Nichols Algebra \( N \)

**generators:** \( g, x_i \ (i = 1, \ldots, n) \)

**relations:** \( g^2 = 1, \ x_i x_j = -x_j x_i, \ g x_i = x_i g \)

\[
\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1} = g
\]

\[
\Delta(x_i) = x_i \otimes g + 1 \otimes x_i, \quad \epsilon(x_i) = 0, \quad S(x_i) = -x_i g
\]

\[\text{Aut}(N) = \text{GL}_n(\mathbb{F}), \quad a = (a_{i,j}) \in \text{GL}_n(\mathbb{F}) \text{ where}\]

\[
\phi_a(g) = g, \quad \phi_a(x_i) = \sum_j a_{i,j} x_j
\]

\( \sigma, \rho \) block diagonal w.r.t. \( \{x_i \mid i = 1, \ldots, n\} \) with blocks

\[
\sigma : \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \quad \rho : \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad \omega^3 = 1.
\]

**Thm.** (Madariaga) \( N \) is a non-cocomm. Hopf alg. with triality.