Pattern Avoidance in Latin Squares

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Outline

- Pattern Avoidance in Permutations
- Pattern Avoidance in Latin Squares
- Future Work and Open Questions

Notation: $\pi = 6371254$.

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Definition

A permutation $\pi \in S_n$ *contains* a permutation $\sigma \in S_m$ if there is a subsequence of π order-isomorphic to σ. If π does not contain σ, then π *avoids* σ.

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Example: $\pi = 6371254$ contains 321, but avoids 1234.

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Wilf-equivalence is an equivalence relation.

Theorem (Simion, Schmidt - 1985)

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$$|Av_n(\pi)| = \frac{1}{n+1}\binom{2n}{n} \textit{ for all } \pi \in S_3.$$

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$Av_n(1324)$ is open!!

Convention: Latin squares of order n are filled with the numbers $1, \ldots, n$.

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Definition

A Latin square L of order n *contains* the permutation π if any row or column of L contains π when read left-to-right or top-to-bottom. L *avoids* π if L does not contain π.

Examples:

1	2	3
2	3	1
3	1	2

1	2	3
3	1	2
2	3	1

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1	2	3
2	3	1
3	1	2

1	2	3
3	1	2
2	3	1

4	3	2	1
3	2	1	4
2	1	4	3
1	4	3	2

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

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$$AvL_3(123) = \left\{ \begin{array}{|c|c|c|} \hline 3 & 2 & 1 \\ \hline 2 & 1 & 3 \\ \hline 1 & 3 & 2 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 2 & 1 & 3 \\ \hline 1 & 3 & 2 \\ \hline 3 & 2 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 3 & 2 \\ \hline 2 & 1 & 3 \\ \hline 3 & 2 & 1 \\ \hline \end{array} \right\}$$

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Theorem

The number of Latin squares of order n row (column) avoiding $\pi \in S_3$ is $n!$.

Flavor of proof:

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For any permutation $\pi \in S_n$, there is exactly one Latin square avoiding 123 in the columns with π as its first row.

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3	2	4	1
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1	4	2	3
4	3	1	2

Column-avoids 123.

Theorem

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Theorem

$AvL_n(123) = AvL_n(231) = AvL_n(312)$ *and*
$AvL_n(132) = AvL_n(213) = AvL_n(321)$.

What about larger patterns?

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Theorem

For a fixed n and any $\pi, \sigma \in S_n$, $|AvL_n(\pi)| = |AvL_n(\sigma)|$.

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Thank You!