Magic squares of Lie groups

Tevian Dray[1], Joshua Kincaid[1], John Huerta[2],
Corinne Manogue[3], Aaron Wangberg[4], Robert Wilson[5]

[1]Department of Mathematics
Oregon State University

[2]Centro de Análise Matemática,
Geometria e Sistemas Dinâmicos
Instituto Superior Técnico (Lisboa)

[3]Department of Physics
Oregon State University

[4]Dept of Mathematics & Statistics
Winona State University

[5]School of Mathematical Sciences
Queen Mary, University of London

(supported by FQXi and the John Templeton Foundation)

The Freudenthal-Tits Magic Square

Freudenthal (1964), Tits (1966):

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{a}_1$	$\mathfrak{a}_2$	$\mathfrak{c}_3$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{a}_2$	$\mathfrak{a}_2 \oplus \mathfrak{a}_2$	$\mathfrak{a}_5$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{c}_3$	$\mathfrak{a}_5$	$\mathfrak{d}_6$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

Vinberg (1966):

$$sa(3, \mathbb{A} \otimes \mathbb{B}) \oplus \mathrm{der}(\mathbb{A}) \oplus \mathrm{der}(\mathbb{B})$$

$$\mathrm{der}(\mathbb{H}) = \mathfrak{so}(3); \quad \mathrm{der}(\mathbb{O}) = \mathfrak{g}_2$$

Goal: Group description: "$SU(3, \mathbb{A} \otimes \mathbb{B})$"

History

- Barton & Sudbery (2003):
  Well-understood in terms of Lie algebras.
- Satisfactory group description not yet known.
- Rosenfeld (1956/1997):
  Isometry groups of projective planes over $\mathbb{A} \otimes \mathbb{B}$.

  Cayley-Moufang plane: $F_4 \longleftrightarrow \mathbb{OP}^2$

- Baez (2002):
  OK for E_6; not for E_7, E_8.

  In short, more work must be done before we can claim to fully understand the geometrical meaning of the Lie groups E_6, E_7 and E_8.

2 × 2:

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{d}_1$	$\mathfrak{a}_1$	$\mathfrak{b}_2$	$\mathfrak{b}_4$
$\mathbb{C}$	$\mathfrak{a}_1$	$\mathfrak{a}_1 \oplus \mathfrak{a}_1$	$\mathfrak{d}_3$	$\mathfrak{d}_5$
$\mathbb{H}$	$\mathfrak{b}_2$	$\mathfrak{d}_3$	$\mathfrak{d}_4$	$\mathfrak{d}_6$
$\mathbb{O}$	$\mathfrak{b}_4$	$\mathfrak{d}_5$	$\mathfrak{d}_6$	$\mathfrak{d}_8$

1	3	10	36
3	6	15	45
10	15	28	66
36	45	66	120

3 × 3:

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{a}_1$	$\mathfrak{a}_2$	$\mathfrak{c}_3$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{a}_2$	$\mathfrak{a}_2 \oplus \mathfrak{a}_2$	$\mathfrak{a}_5$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{c}_3$	$\mathfrak{a}_5$	$\mathfrak{d}_6$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

3	8	21	52
8	16	35	78
21	35	66	133
52	45	133	248

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	SO(2)	SO(3)	SO(5)	SO(9)
$\mathbb{C}'$	SO(2, 1)	SO(3, 1)	SO(5, 1)	SO(9, 1)
$\mathbb{H}'$	SO(3, 2)	SO(4, 2)	SO(6, 2)	SO(10, 2)
$\mathbb{O}'$	SO(5, 4)	SO(6, 4)	SO(8, 4)	SO(12, 4)

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathrm{SU}(2,\mathbb{R})$	$\mathrm{SU}(2,\mathbb{C})$	$\mathrm{SU}(2,\mathbb{H})$	$\mathrm{SU}(2,\mathbb{O})$
$\mathbb{C}'$	$\mathrm{SL}(2,\mathbb{R})$	$\mathrm{SL}(2,\mathbb{C})$	$\mathrm{SL}(2,\mathbb{H})$	$\mathrm{SL}(2,\mathbb{O})$
$\mathbb{H}'$	$\mathrm{Sp}(4,\mathbb{R})$	$\mathrm{SU}(2,2,\mathbb{C})$		
$\mathbb{O}'$				

$$d = 3, 4, 6, 10$$

(1980s: Corrigan,Evans,Fairlie,Manogue,Sudbery)

$$|\mathrm{Sp}(4,\mathbb{R})| = 10;\ |\mathrm{SU}(2,2,\mathbb{C})| = 15$$

But: $|\mathrm{SU}(2,2,\mathbb{R})| = 6;\ |\mathrm{Sp}(4,\mathbb{C})| = 20$

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	SO(2)	SO(3)	SO(5)	SO(9)
$\mathbb{C}'$	SO(2, 1)	SO(3, 1)	SO(5, 1)	SO(9, 1)
$\mathbb{H}'$	SO(3, 2)	SO(4, 2)	SO(6, 2)	SO(10, 2)
$\mathbb{O}'$	SO(5, 4)	SO(6, 4)	SO(8, 4)	SO(12, 4)

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$SU(2,\mathbb{R})$	$SU(2,\mathbb{C})$	$SU(2,\mathbb{H})$	$SU(2,\mathbb{O})$
$\mathbb{C}'$	$SL(2,\mathbb{R})$	$SL(2,\mathbb{C})$	$SL(2,\mathbb{H})$	$SL(2,\mathbb{O})$
$\mathbb{H}'$	$Sp(4,\mathbb{R})$	$Sp(4,\mathbb{C})$	$Sp(4,\mathbb{H})$	$Sp(4,\mathbb{O})$
$\mathbb{O}'$	??	??	??	??

$$d = 3, 4, 6, 10$$

(1980s: Corrigan,Evans,Fairlie,Manogue,Sudbery)

$$|Sp(4,\mathbb{R})| = 10;\ |SU(2,2,\mathbb{C})| = 15$$

But: $|SU(2,2,\mathbb{R})| = 6;\ |Sp(4,\mathbb{C})| = 20$

Symplectic Groups

Definition

$$\mathrm{Sp}(2k,\mathbb{R}) = \{Q \in M^{2k\times 2k}(\mathbb{R}) : Q\Omega Q^T = \Omega\}$$

$$\mathrm{Sp}(4,\mathbb{R}) = \mathrm{SO}(3,2) \qquad |\mathrm{Sp}(4,\mathbb{R})| = 10$$

$$\mathrm{Sp}(6,\mathbb{R}) = \mathrm{C}_3 \text{ (split)} \qquad |\mathrm{Sp}(6,\mathbb{R})| = 21$$

$$\Omega = \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)$$

Symplectic Groups

Definition (traditional)

$$\mathrm{Sp}(2k,\mathbb{K}) = \{Q \in M^{2k\times 2k}(\mathbb{K}) : Q\Omega Q^T = \Omega\}$$

$$\mathrm{Sp}(4,\mathbb{C}) = C_2 \text{ (complex)} \qquad |\mathrm{Sp}(4,\mathbb{C})| = 20$$

$$\mathrm{Sp}(6,\mathbb{C}) = C_3 \text{ (complex)} \qquad |\mathrm{Sp}(6,\mathbb{C})| = 42$$

$$\Omega = \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)$$

Symplectic Groups

Definition (Sudbery)

$$\mathrm{Sp}(2k,\mathbb{K}) = \{Q \in M^{2k\times 2k}(\mathbb{K}) : Q\Omega Q^\dagger = \Omega\}$$

Fine Print: Count "phases" separately

$$\mathrm{Sp}(4,\mathbb{C}) = \mathrm{SU}(2,2) \qquad |\mathrm{Sp}(4,\mathbb{C})| = 15$$
$$\mathrm{Sp}(6,\mathbb{C}) = \mathrm{SU}(3,3) \qquad |\mathrm{Sp}(6,\mathbb{C})| = 35$$

$$\Omega = \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)$$

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathrm{SU}(3,\mathbb{R})$	$\mathrm{SU}(3,\mathbb{C})$	$\mathrm{SU}(3,\mathbb{H})$	F_4
$\mathbb{C}'$	$\mathrm{SL}(3,\mathbb{R})$	$\mathrm{SL}(3,\mathbb{C})$	$\mathrm{SL}(3,\mathbb{H})$	$\mathrm{E}_{6(-26)}$
$\mathbb{H}'$	$\mathrm{Sp}(6,\mathbb{R})$	$\mathrm{SU}(3,3,\mathbb{C})$	$\mathrm{D}_{6(-6)}$	$\mathrm{E}_{7(-25)}$
$\mathbb{O}'$	$\mathrm{F}_{4(4)}$	$\mathrm{E}_{6(2)}$	$\mathrm{E}_{7(-5)}$	$\mathrm{E}_{8(-24)}$

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathrm{SU}(3,\mathbb{R})$	$\mathrm{SU}(3,\mathbb{C})$	$\mathrm{SU}(3,\mathbb{H})$	$\mathrm{SU}(3,\mathbb{O})$
$\mathbb{C}'$	$\mathrm{SL}(3,\mathbb{R})$	$\mathrm{SL}(3,\mathbb{C})$	$\mathrm{SL}(3,\mathbb{H})$	$\mathrm{SL}(3,\mathbb{O})$
$\mathbb{H}'$	$\mathrm{Sp}(6,\mathbb{R})$	$\mathrm{SU}(3,3,\mathbb{C})$	$\mathrm{D}_{6(-6)}$	$\mathrm{E}_{7(-25)}$
$\mathbb{O}'$	$\mathrm{F}_{4(4)}$	$\mathrm{E}_{6(2)}$	$\mathrm{E}_{7(-5)}$	$\mathrm{E}_{8(-24)}$

Dray & Manogue (2010):

$F_4 \cong \mathrm{SU}(3,\mathbb{O})$, $E_6 \cong \mathrm{SL}(3,\mathbb{O})$ using $\mathrm{SL}(2,\mathbb{O}) \cong Spin(9,1) \subset E_6$

$$\mathcal{X} = \begin{pmatrix} X & \theta \\ \theta^\dagger & n \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$$

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathrm{SU}(3,\mathbb{R})$	$\mathrm{SU}(3,\mathbb{C})$	$\mathrm{SU}(3,\mathbb{H})$	$\mathrm{SU}(3,\mathbb{O})$
$\mathbb{C}'$	$\mathrm{SL}(3,\mathbb{R})$	$\mathrm{SL}(3,\mathbb{C})$	$\mathrm{SL}(3,\mathbb{H})$	$\mathrm{SL}(3,\mathbb{O})$
$\mathbb{H}'$	$\mathrm{Sp}(6,\mathbb{R})$	$\mathrm{Sp}(6,\mathbb{C})$	$\mathrm{Sp}(6,\mathbb{H})$	$\mathrm{Sp}(6,\mathbb{O})$
$\mathbb{O}'$	??	??	??	??

Dray & Manogue (2010):

$F_4 \cong \mathrm{SU}(3,\mathbb{O})$, $E_6 \cong \mathrm{SL}(3,\mathbb{O})$ using $\mathrm{SL}(2,\mathbb{O}) \cong Spin(9,1) \subset E_6$

$$\mathcal{X} = \begin{pmatrix} X & \theta \\ \theta^\dagger & n \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$$

Conformalization

SO(4, 2) is the "conformalization" of SO(3, 1)

$$\text{conformal} = \text{Lorentz} + \text{translations} + \text{conformal} + \text{dilation}$$
$$15 = 6 + 4 + 4 + 1$$

$$\text{acts on } \mathbf{P} = \begin{pmatrix} p+q & X \\ -\widetilde{X} & p-q \end{pmatrix} \longleftrightarrow \{X, p, q\}$$

$E_{7(-25)}$ is the "conformalization" of $E_{6(-26)}$

$$\text{conformal} = \text{group} + (2 \times \text{null rotations}) + \text{phase}$$
$$133 = 78 + (2 \times 27) + 1$$

acts on Freudenthal triple: $\{\mathcal{X}, \mathcal{Y}, p, q\}$

SU(3, 3)

$\mathbb{C}'$	$\mathrm{SL}(3,\mathbb{R})$	$\mathrm{SL}(3,\mathbb{C})$	$\mathrm{SL}(3,\mathbb{H})$	$E_{6(-26)}$
$\mathbb{H}'$	$Sp(3,\mathbb{R})$	$\mathrm{SU}(3,3,\mathbb{C})$	$D_6(III)$	$E_{7(-25)}$

$\mathrm{SU}(3,3,\mathbb{C})$ is the conformal group corresponding to $\mathrm{SL}(3,\mathbb{C})$.

$$\text{conformal} = \text{group} + (2 \times \text{null rotations}) + \text{phase}$$
$$35 = 16 + (2 \times 9) + 1$$

$$\{\mathcal{X}, \mathcal{Y}, r, s\} : \qquad 20 = 2 \times 9 + 2$$

20-dimensional representation of SU(6): 3-forms in 6 dimensions

Squares and Cubes

2-forms in d = 4: $(6 = 1 + 4 + 1)$

00	01
10	11

$$00 \longleftrightarrow r$$
$$01, 10 \longleftrightarrow X$$
$$11 \longleftrightarrow s$$

3-forms in d = 6: $(20 = 1 + 9 + 9 + 1)$

$$000 \longleftrightarrow r$$
$$001, 010, 100 \longleftrightarrow \mathcal{X}$$
$$011, 110, 101 \longleftrightarrow \mathcal{Y}$$
$$111 \longleftrightarrow s$$

From $SU(2,2,\mathbb{C})$ to $Sp(4,\mathbb{C})$

6-dim representation of SU(4): 2-forms in 4-d: $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\widetilde{X}\epsilon = (X\epsilon)^T \Longrightarrow \begin{pmatrix} r & X \\ -\widetilde{X} & s \end{pmatrix} \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \text{ antisymmetric}$$

$$\mathbf{M}[u \wedge v] = \mathbf{M}u \wedge \mathbf{M}v$$

$$\mathbf{P}^T = -\mathbf{P} \Longrightarrow \mathbf{P} \longmapsto \mathbf{M}\mathbf{P}\mathbf{M}^T$$

BUT: need *irreducible* representations of Sp(4):

$$4 \wedge 4 = \begin{cases} 5 \oplus 1 \text{ as representations of } Sp(4,\mathbb{R}) \\ 6 \oplus 6 \text{ as representations of } Sp(4,\mathbb{C}) \end{cases}$$

From $\mathrm{SU}(3,3,\mathbb{C})$ to $\mathrm{Sp}(6,\mathbb{C})$

20-dim representation of SU(6): 3-forms in 6-d:

$$\dot{\mathbf{M}}[u \wedge v \wedge w] = \dot{\mathbf{M}}u \wedge v \wedge w + u \wedge \dot{\mathbf{M}}v \wedge w + u \wedge v \wedge \dot{\mathbf{M}}w$$

$$\dot{\mathbf{M}} = \begin{pmatrix} \phi + \rho & \mathcal{A} \\ \mathcal{B} & \phi' - \rho \end{pmatrix}$$

$$\begin{aligned} \mathcal{X} &\longmapsto \phi(\mathcal{X}) + \frac{1}{3}\rho\mathcal{X} + 2\mathcal{B} * \mathcal{Y} + \mathcal{A}s \\ \mathcal{Y} &\longmapsto 2\mathcal{A} * \mathcal{X} + \phi'(\mathcal{Y}) - \frac{1}{3}\rho\mathcal{Y} + \mathcal{B}r \\ r &\longmapsto \mathrm{tr}(\mathcal{A} \circ \mathcal{Y}) - \rho r \\ s &\longmapsto \mathrm{tr}(\mathcal{B} \circ \mathcal{X}) + \rho s \end{aligned}$$

BUT: need *irreducible* (real!) representations of Sp(6):

$$6 \wedge 6 \wedge 6 = \begin{cases} 14 \oplus 6 \text{ as representations of } \mathrm{Sp}(6,\mathbb{R}) \\ 20 \oplus 18 \text{ as representations of } \mathrm{Sp}(6,\mathbb{C}) \end{cases}$$

$\mathrm{Sp}(6, \mathbb{H})$?

Recall: $\dot{\mathbf{M}}[u \wedge v \wedge w] = \dot{\mathbf{M}}u \wedge v \wedge w + u \wedge \dot{\mathbf{M}}v \wedge w + u \wedge v \wedge \dot{\mathbf{M}}w$

- Not well defined over $\mathbb{H}$!
- No way (yet...) to fix ordering.
- Use Freudenthal to *define* action.
- Works — but is this still *cubies*?
- Get: $E_7 \cong \mathrm{Sp}(6, \mathbb{O})$?

The Subgroup Structure of E_6

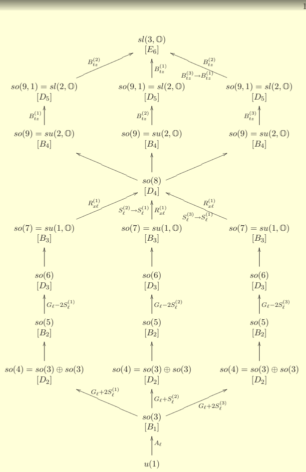

Figure 4.2: Chain of subgroups $SO(n) \subset SO(9,1) \subset SL(3,\mathbb{O})$

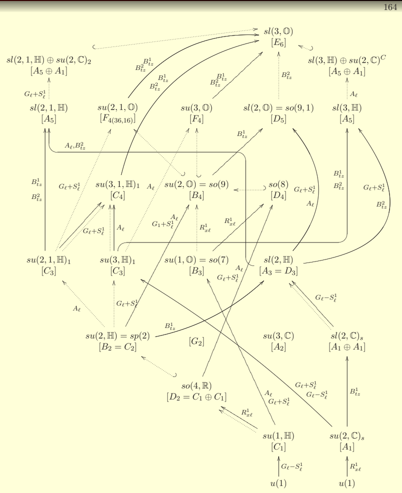

Figure 5.1: Preferred subalgebra chains of E_6 using the same basis

Wangberg (PhD 2007), Wangberg & Dray (2012 preprint)

Cartan Decompositions of E_6

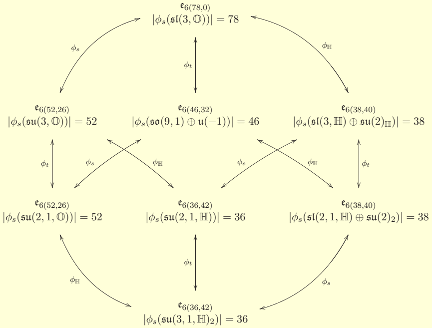

FIG. 3. Composition of associated Cartan maps of $\mathfrak{e}_6$ acting on real forms of $\mathfrak{e}_6$, showing the maximal compact subalgebra under ϕ_s.

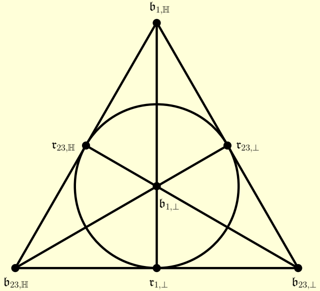

FIG. 4. Composition of associated Cartan maps of E_6 acting on real forms of E_6, showing the maximal compact subalgebra under ϕ_s.

Wangberg (PhD 2007), Wangberg & Dray (JMP 2013)

Generating Rotation Groups

Clifford algebra: $\{\Gamma_\nu, \Gamma_\mu\} = \Gamma_\nu \Gamma_\mu + \Gamma_\mu \Gamma_\nu = 2 g_{\nu\mu} I$

- Real vector space; $P = x^\nu \Gamma_\nu$
- Generate special orthogonal group

$$R_{\nu\mu} = \exp[\Gamma_\nu \Gamma_\mu \theta/2] = \begin{cases} I \cos\alpha + \Gamma_\nu \Gamma_\mu \sin\alpha, & (\Gamma_\nu \Gamma_\mu)^2 = -I \\ I \cosh\alpha + \Gamma_\nu \Gamma_\mu \sinh\alpha, & (\Gamma_\nu \Gamma_\mu)^2 = I \end{cases}$$

$$P \longmapsto R_{\nu\mu} P R_{\nu\mu}^{-1}$$

"Rotation" in $\nu\mu$-plane: $\mathrm{SO}(n, m)$

Matrices over $\mathbb{H}' \otimes \mathbb{C}$

$$X = \begin{pmatrix} A & \overline{a} \\ a & -A^* \end{pmatrix} = x^\mu \sigma_\mu \qquad (A \in \mathbb{H}', a \in \mathbb{C})$$

Trace reversal: $\widetilde{X} = X - \mathrm{tr}(X) I = \begin{pmatrix} A^* & \overline{a} \\ a & -A \end{pmatrix}$

$$P = \begin{pmatrix} 0 & X \\ \widetilde{X} & 0 \end{pmatrix} = x^\mu \Gamma_\mu$$

$$\longmapsto \mathrm{SO}(4,2) \cong \mathrm{SU}(2,2,\mathbb{C}) \cong \mathrm{SU}(2, \mathbb{H}' \otimes \mathbb{C})$$

[Kincaid (MS 2012)]

Matrices over $\mathbb{K}' \otimes \mathbb{K}$

$$X = \begin{pmatrix} A & \overline{a} \\ a & -A^* \end{pmatrix} = x^\mu \sigma_\mu \qquad (A \in \mathbb{K}', a \in \mathbb{K})$$

$$P = \begin{pmatrix} 0 & X \\ \widetilde{X} & 0 \end{pmatrix} = x^\mu \Gamma_\mu$$

$$\longmapsto \mathrm{SO}(k', k) \cong \mathrm{SU}(2, \mathbb{K}' \otimes \mathbb{K})$$

[Dray, Huerta, & Kincaid (in preparation)]

The Fourth Row

$\mathbb{O}'$	$\mathrm{F}_{4(4)}$	$\mathrm{E}_{6(2)}$	$\mathrm{E}_{7(-5)}$	$\mathrm{E}_{8(-24)}$

- Metasymplectic geometry? (Freudenthal 1959)
- Adjoint representation!
- Start with E_8.
  Get adjoint representations throughout both magic squares.
- Also get minimal representations!
- (Wilson, 2013) Explicit representation of E_8 (not $\mathfrak{e}_8$) in terms of 3×3 "matrices" over $\mathbb{O}' \otimes \mathbb{O}$.

SUMMARY

- **Have:** $E_6 \cong \mathrm{SL}(3, \mathbb{O})$
  [Dray & Manogue (2010)]
- **Have:** Structure of E_6
  [Wangberg (PhD 2007), Wangberg & Dray (2012, 2013)]
- **Have:** 2×2 Magic Square as $\mathrm{SU}(2, \mathbb{K}' \otimes \mathbb{K})$
  [Kincaid (MS 2012), Kincaid, Dray, & Huerta (in preparation)]
- **(Mostly) Have:** $E_7 \cong \mathrm{Sp}(6, \mathbb{O})$
  [Dray, Huerta, Manogue, Wilson (in progress)]
- **Should Have:** $E_7 \cong \mathrm{SU}(3, \mathbb{H}' \otimes \mathbb{O})$
- **Might have:** $E_8 \cong \mathrm{SU}(3, \mathbb{O}' \otimes \mathbb{O})$