A class of latin squares derived from finite abelian groups

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Abstract

We consider latin squares obtained by extending the Cayley tables of finite abelian groups, and give preliminary results on the existence/nonexistence of latin squares orthogonal to these.

Extending the Cayley table of $\mathbb{Z}_6$.

The Cayley table of $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$.

	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Extending the Cayley table of $\mathbb{Z}_6$.

The Cayley table of $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$.

Extend the symbol set to $\{0, 1, 2, 3, 4, 5, a\}$.

	0	1	2	3	4	5	a
0	0	1	2	3	4	5	a
1	1	2	3	4	5	0	
2	2	3	4	5	0	1	
3	3	4	5	0	1	2	
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a	a						

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3	3	4	5	0	1	2	2
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2	2	3	4	5	a	1	0
3	3	4	5	0	1	a	2
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The general construction.

$G = \{g_0, \ldots, g_{m-1}\}$, $g_0 = 0$, an abelian group.

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Form a latin square, $Ext_\theta(G; a)$.

$$Ext_\theta(G; a) = \left(\begin{array}{c|ccc|c} g_0 & g_1 & \cdots & g_{m-1} & a \\ \hline g_1 & & & & \\ \vdots & & A & & B \\ g_{m-1} & & & & \\ \hline a & & C & & w \end{array}\right)$$

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The $g_i g_j$th entry of A is $\begin{cases} a & \text{if } \theta(g_i) = g_j, \\ g_i + g_j & \text{if } \theta(g_i) \neq g_j. \end{cases}$

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The ith entry of B is $g_i + \theta(g_i)$.

The jth entry of C is $g_j + \theta^{-1}(g_j)$.

Characterizing θ and w.

Define η by $\eta(g_i) = \theta(g_i) + g_i$.

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Lemma

- If G has a unique involution δ, then
  - $w = \delta$, and
  - θ is a near complete mapping of G, i.e., η is a bijection $\{g_1, \dots, g_{m-1}\} \to \{g_0, \dots, g_{m-1}\} \setminus \{\delta\}$.

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- Otherwise
  - $w = 0$, and
  - θ is a "complete mapping" of G, i.e., η is a bijection $\{g_1, \ldots, g_{m-1}\} \to \{g_1, \ldots, g_{m-1}\}$.

A transversal in $Ext(\mathbb{Z}_6; a)$

$$\begin{pmatrix} 0 & 1 & \color{red}{2} & 3 & 4 & 5 & a \\ 1 & 2 & 3 & a & 5 & 0 & \color{red}{4} \\ 2 & 3 & 4 & \color{red}{5} & a & 1 & 0 \\ 3 & 4 & 5 & 0 & \color{red}{1} & a & 2 \\ 4 & a & 0 & 1 & 2 & \color{red}{3} & 5 \\ 5 & \color{red}{0} & a & 2 & 3 & 4 & 1 \\ \color{red}{a} & 5 & 1 & 4 & 0 & 2 & 3 \end{pmatrix}$$

a) Each row contains one cell of the transversal.

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a) Each row contains one cell of the transversal.

b) Each column contains one cell of the transversal.

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a) Each row contains one cell of the transversal.
b) Each column contains one cell of the transversal.
c) Each symbol appears exactly once in the transversal.

Deviations and the Δ- lemma.

Let L be a latin square with rows and columns indexed by the elements $\{g_0, \ldots, g_{m-1}\}$ of an abelian group G.

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If cell C is in row g_i and column g_j, and its entry is g_k, then

$$dev(C) = g_k - (g_i + g_j).$$

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$$dev(C) = g_k - (g_i + g_j).$$

The Δ-lemma

Let $C_1, \ldots, C_m$ be the cells of a transversal of L.

- If G has a *unique involution* δ, then

$$\sum_{i=1}^{m} dev(C_i) = \delta.$$

- Otherwise

$$\sum_{i=1}^{m} dev(C_i) = 0.$$

Orthogonal latin squares.

A pair of orthogonal latin squares of order 5.

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 2 & 4 & 1 & 3 \\ 1 & 3 & 0 & 2 & 4 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 0 & 2 & 4 & 1 \\ 4 & 1 & 3 & 0 & 2 \end{pmatrix}$$

$$\searrow \text{superimposed} \swarrow$$

$$\begin{pmatrix} 0,0 & 1,2 & 2,4 & 3,1 & 4,3 \\ 1,1 & 2,3 & 3,0 & 4,2 & 0,4 \\ 2,2 & 3,4 & 4,1 & 0,3 & 1,0 \\ 3,3 & 4,0 & 0,2 & 1,4 & 2,1 \\ 4,4 & 0,1 & 1,3 & 2,0 & 3,2 \end{pmatrix}$$

Each ordered pair of symbols appears exactly once.

Orthogonality and transversals.

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$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \color{red}{0} & 2 & 4 & 1 & 3 \\ 1 & 3 & \color{red}{0} & 2 & 4 \\ 2 & 4 & 1 & 3 & \color{red}{0} \\ 3 & \color{red}{0} & 2 & 4 & 1 \\ 4 & 1 & 3 & \color{red}{0} & 2 \end{pmatrix}$$

The red entries in the second square are all 0.

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The corresponding entries in the first square form a transversal.

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The red entries in the second square are all 0.

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Lemma

A latin square has an orthogonal mate if and only its set of cells can be partitioned by some set of transversals.

Some new classes of confirmed bachelor squares.

Definition

A bachelor square is a latin square without an orthogonal mate.

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Theorem

If G does not have a unique involution, then $Ext(G; a)$ is a confirmed bachelor square.

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Examples

- If $m \equiv 0 \pmod 4$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_{m/2}$, then $Ext(G; a)$ is a confirmed bachelor square of order $m + 1$.

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Examples

- If $m \equiv 0 \pmod 4$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_{m/2}$, then $Ext(G; a)$ is a confirmed bachelor square of order $m + 1$.
- If m is odd and $G = \mathbb{Z}_m$, then $Ext(G; a)$ is a confirmed bachelor square of order $m + 1$.

Proof by example.

$$Ext(\mathbb{Z}_7; a) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & a \\ 1 & a & 3 & 4 & 5 & 6 & 0 & 2 \\ 2 & 3 & a & 5 & 6 & 0 & 1 & 4 \\ 3 & 4 & 5 & a & 0 & 1 & 2 & 6 \\ 4 & 5 & 6 & 0 & a & 2 & 3 & 1 \\ 5 & 6 & 0 & 1 & 2 & a & 4 & 3 \\ 6 & 0 & 1 & 2 & 3 & 4 & a & 5 \\ a & 2 & 4 & 6 & 1 & 3 & 5 & 0 \end{pmatrix}$$

Suppose there is a transversal through the "a" in the last row.

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Suppose there is a transversal through the "a" in the last row.
None of the red entries can be on this transversal.

Proof by example.

$$Ext(\mathbb{Z}_7; a) = \begin{pmatrix} & 1 & 2 & 3 & 4 & 5 & 6 & \\ & & 3 & 4 & 5 & 6 & 0 & 2 \\ & 3 & & 5 & 6 & 0 & 1 & 4 \\ & 4 & 5 & & 0 & 1 & 2 & 6 \\ & 5 & 6 & 0 & & 2 & 3 & 1 \\ & 6 & 0 & 1 & 2 & & 4 & 3 \\ & 0 & 1 & 2 & 3 & 4 & & 5 \\ a & & & & & & & \end{pmatrix}$$

Suppose there is a transversal through the "a" in the last row.
None of the red entries can be on this transversal. Remove these.

Proof by example.

$$
Ext(\mathbb{Z}_7; a) = \left(\begin{array}{cccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & \\
 & & 3 & 4 & 5 & 6 & 0 & \color{red}{2} \\
 & 3 & & 5 & 6 & 0 & 1 & \color{red}{4} \\
 & 4 & 5 & & 0 & 1 & 2 & \color{red}{6} \\
 & 5 & 6 & 0 & & 2 & 3 & \color{red}{1} \\
 & 6 & 0 & 1 & 2 & & 4 & \color{red}{3} \\
 & 0 & 1 & 2 & 3 & 4 & & \color{red}{5} \\
\color{blue}{a} & & & & & & &
\end{array} \right)
$$

Suppose there is a transversal through the "a" in the last row.
None of the red entries can be on this transversal. Remove these.
Rearrange columns: move "red" column.

Proof by example.

$$Ext(\mathbb{Z}_7; a) = \begin{pmatrix} & & 1 & 2 & 3 & 4 & 5 & 6 \\ & \color{red}{2} & & 3 & 4 & 5 & 6 & 0 \\ & \color{red}{4} & 3 & & 5 & 6 & 0 & 1 \\ & \color{red}{6} & 4 & 5 & & 0 & 1 & 2 \\ & \color{red}{1} & 5 & 6 & 0 & & 2 & 3 \\ & \color{red}{3} & 6 & 0 & 1 & 2 & & 4 \\ & \color{red}{5} & 0 & 1 & 2 & 3 & 4 & \\ \color{blue}{a} & & & & & & & \end{pmatrix}$$

Suppose there is a transversal through the "a" in the last row.
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Suppose there is a transversal through the "a" in the last row.
None of the red entries can be on this transversal. Remove these.
Rearrange columns: move "red" column.
Compute deviations.

Proof by example.

$$Ext(\mathbb{Z}_7; a) = \begin{pmatrix} & & 0 & 0 & 0 & 0 & 0 & 0 \\ & \color{red}{1} & & 0 & 0 & 0 & 0 & 0 \\ & \color{red}{2} & 0 & & 0 & 0 & 0 & 0 \\ & \color{red}{3} & 0 & 0 & & 0 & 0 & 0 \\ & \color{red}{4} & 0 & 0 & 0 & & 0 & 0 \\ & \color{red}{5} & 0 & 0 & 0 & 0 & & 0 \\ & \color{red}{6} & 0 & 0 & 0 & 0 & 0 & \\ \color{blue}{a} & & & & & & & \end{pmatrix}$$

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Suppose there is a transversal through the "a" in the last row.
None of the red entries can be on this transversal. Remove these.
Rearrange columns: move "red" column.
Compute deviations.
The transversal must contain exactly one cell from each column and the deviations must add to 0.

Proof by example.

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Suppose there is a transversal through the "a" in the last row.
None of the red entries can be on this transversal. Remove these.
Rearrange columns: move "red" column.
Compute deviations.
The transversal must contain exactly one cell from each column and the deviations must add to 0.
This is impossible: a contradiction.

Some bachelor/monogamous squares?

Definition

A monogamous square is a latin square that has an orthogonal mate, but is not contained in a set of three pairwise orthogonal latin squares.

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Theorem

If G has a unique involution, then $Ext_{\theta}(G; a)$ is a either a bachelor square or a monogamous square.

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Question

For which θ is $Ext_\theta(G;a)$ a bachelor square; for which a monogamous square?

Some partial transversals.

$$\left(\begin{array}{c|ccccccc|c}
0 & \dots & \theta(\delta) & \dots & \delta & \dots & & \dots & a \\
\hline
\vdots & & \vdots & & \vdots & & \vdots & & \\
\theta^{-1}(\delta) & \dots & & \dots & a & \dots & & \dots & \delta+\theta^{-1}(\delta) \\
\vdots & & \vdots & & & & \vdots & & \\
\delta & \dots & a & & & & & & \\
\vdots & & \vdots & & & & \vdots & & \\
 & & \vdots & & & & a & \dots & 0 \\
\vdots & & \vdots & & & & \vdots & & \\
\hline
\color{red}{a} & & \delta+\theta(\delta) & & & & 0 & & \delta
\end{array}\right)$$

Any transversal through the red a must pass through

Some partial transversals.

$$\left(\begin{array}{c|cccccccc|c}
0 & \dots & \theta(\delta) & \dots & \delta & \dots & & \dots & & a \\
\hline
\vdots & & \vdots & & \vdots & & \vdots & & & \\
\theta^{-1}(\delta) & \dots & & \dots & a & \dots & & \dots & & \delta+\theta^{-1}(\delta) \\
\vdots & & \vdots & & & & \vdots & & & \\
\delta & \dots & a & & & & & & & \\
\vdots & & \vdots & & & & \vdots & & & \\
 & & \vdots & & & & a & \dots & & 0 \\
\vdots & & \vdots & & & & \vdots & & & \\
\hline
a & & \delta+\theta(\delta) & & & & 0 & & & \delta
\end{array}\right)$$

Any transversal through the red a must pass through the red $\delta + \theta^{-1}(\delta)$.

Some partial transversals.

$$
\left(\begin{array}{c|ccccccc|c}
0 & \dots & \theta(\delta) & \dots & \delta & \dots & & \dots & a \\
\hline
\vdots & & \vdots & & \vdots & & \vdots & & \\
\theta^{-1}(\delta) & \dots & & \dots & a & \dots & & \dots & \delta+\theta^{-1}(\delta) \\
\vdots & & \vdots & & & & \vdots & & \\
\delta & \dots & a & & & & & & \\
\vdots & & \vdots & & & & \vdots & & \\
 & & \vdots & & & & a & \dots & 0 \\
\vdots & & \vdots & & & & \vdots & & \\
\hline
a & & \delta+\theta(\delta) & & & & 0 & & \delta
\end{array}\right)
$$

Any transversal through the red a must pass through the red $\delta + \theta^{-1}(\delta)$.
Any transversal through the blue a must pass through

Some partial transversals.

$$
\left(\begin{array}{c|ccccccc|c}
0 & \cdots & \theta(\delta) & \cdots & \delta & \cdots & & \cdots & {\color{blue}a} \\
\hline
\vdots & & \vdots & & \vdots & & \vdots & & \\
\theta^{-1}(\delta) & \cdots & & \cdots & a & \cdots & & \cdots & {\color{red}\delta+\theta^{-1}(\delta)} \\
\vdots & & \vdots & & & & \vdots & & \\
\delta & \cdots & a & & & & & & \\
\vdots & & \vdots & & & & \vdots & & \\
 & & \vdots & & & & a & \cdots & 0 \\
\vdots & & \vdots & & & & \vdots & & \\
\hline
{\color{red}a} & & {\color{blue}\delta+\theta(\delta)} & & & & 0 & & \delta
\end{array}\right)
$$

Any transversal through the red a must pass through the red $\delta + \theta^{-1}(\delta)$.
Any transversal through the blue a must pass through the blue $\delta + \theta(\delta)$.

Some partial transversals.

$$
\left(
\begin{array}{c|ccccccc|c}
0 & \ldots & \theta(\delta) & \ldots & \delta & \ldots & & \ldots & \color{blue}{a} \\
\hline
\vdots & & \vdots & & \vdots & & \vdots & & \\
\theta^{-1}(\delta) & \ldots & & \ldots & a & \ldots & & \ldots & \color{red}{\delta + \theta^{-1}(\delta)} \\
\vdots & & \vdots & & & & \vdots & & \\
\delta & \ldots & a & & & & & & \\
\vdots & & \vdots & & & & \vdots & & \\
 & & \vdots & & & & a & \ldots & 0 \\
\vdots & & \vdots & & & & \vdots & & \\
\hline
\color{red}{a} & & \color{blue}{\delta + \theta(\delta)} & & & & 0 & & \color{green}{\delta}
\end{array}
\right)
$$

Any transversal through the red a must pass through the red $\delta + \theta^{-1}(\delta)$.
Any transversal through the blue a must pass through the blue $\delta + \theta(\delta)$.
Any transversal through the green δ must pass through

Some partial transversals.

$$
\left(\begin{array}{c|ccccccc|c}
0 & \dots & \theta(\delta) & \dots & \delta & \dots & & \dots & \color{blue}{a} \\
\hline
\vdots & & \vdots & & \vdots & & \vdots & & \\
\theta^{-1}(\delta) & \dots & & \dots & a & \dots & & \dots & \color{red}{\delta+\theta^{-1}(\delta)} \\
\vdots & & \vdots & & & & \vdots & & \\
\delta & \dots & a & & & & & & \\
\vdots & & \vdots & & & & \vdots & & \\
 & & \vdots & & & & \color{green}{a} & \dots & 0 \\
\vdots & & \vdots & & & & \vdots & & \\
\hline
\color{red}{a} & & \color{blue}{\delta+\theta(\delta)} & & & & 0 & & \color{green}{\delta}
\end{array}\right)
$$

Any transversal through the red a must pass through the red $\delta + \theta^{-1}(\delta)$.
Any transversal through the blue a must pass through the blue $\delta + \theta(\delta)$.
Any transversal through the green δ must pass through the green a.

Some more partial transversals.

$$\left(\begin{array}{c|ccccc|c} 0 & \dots & \theta(g_s) & \dots & g_j & \dots & a \\ \hline \vdots & & \vdots & & \vdots & & \vdots \\ g_s & \dots & \color{red}{a} & & \vdots & & \vdots \\ \vdots & & & & \vdots & & \vdots \\ g_i & \dots & \dots & \dots & & \dots & \color{red}{g_i + \theta(g_i)} \\ \vdots & & & & \vdots & & \vdots \\ \hline a & \dots & \dots & \dots & \color{red}{g_j + \theta^{-1}(g_j)} & \dots & \delta \end{array}\right)$$

If the three red entries are on the same transversal, then

Some more partial transversals.

$$\left(\begin{array}{c|ccccc|c}
0 & \dots & \theta(g_s) & \dots & g_j & \dots & a \\
\hline
\vdots & & \vdots & & \vdots & & \vdots \\
g_s & \dots & \color{red}{a} & & \vdots & & \vdots \\
\vdots & & & & \vdots & & \vdots \\
g_i & \dots & \dots & \dots & & \dots & \color{red}{g_i + \theta(g_i)} \\
\vdots & & & & \vdots & & \vdots \\
\hline
a & \dots & \dots & \dots & \color{red}{g_j + \theta^{-1}(g_j)} & \dots & \delta
\end{array}\right)$$

If the three red entries are on the same transversal, then

$$\theta(g_i) + \theta^{-1}(g_j) = g_s + \theta(g_s) + \delta.$$

Some more partial transversals.

$$\left(\begin{array}{c|ccccc|c}
0 & \dots & \theta(g_s) & \dots & g_j & \dots & a \\
\hline
\vdots & & \vdots & & \vdots & & \vdots \\
g_s & \dots & {\color{red}a} & & \vdots & & \vdots \\
\vdots & & & & \vdots & & \vdots \\
g_i & \dots & \dots & \dots & & \dots & {\color{red}g_i + \theta(g_i)} \\
\vdots & & & & \vdots & & \vdots \\
\hline
a & \dots & \dots & \dots & {\color{red}g_j + \theta^{-1}(g_j)} & \dots & \delta
\end{array}\right)$$

If the three red entries are on the same transversal, then

$$\theta(g_i) + \theta^{-1}(g_j) = g_s + \theta(g_s) + \delta.$$

$$\theta(g_i), \theta^{-1}(g_j), g_s + \theta(g_s) + \delta \in \{g_0, \dots, g_{m-1}\} \setminus \{0, \delta\}.$$

An example: $Ext(\mathbb{Z}_4; a)$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & a \\ 1 & 2 & a & 0 & 3 \\ 2 & 3 & 0 & a & 1 \\ 3 & a & 1 & 2 & 0 \\ a & 0 & 3 & 1 & 2 \end{pmatrix}$$

An example: $Ext(\mathbb{Z}_4; a)$

$$\left(\begin{array}{c|ccc|c} 0 & 1 & 2 & 3 & a \\ \hline 1 & 2 & a & 0 & 3 \\ 2 & 3 & 0 & a & 1 \\ 3 & a & 1 & 2 & 0 \\ \hline a & 0 & 3 & 1 & 2 \end{array}\right)$$

$$\theta(g_i), \theta^{-1}(g_j), g_s + \theta(g_s) + \delta \in \{1, 3\}.$$

An example: $Ext(\mathbb{Z}_4; a)$

$$\left(\begin{array}{c|ccc|c} 0 & 1 & 2 & 3 & a \\ \hline 1 & 2 & a & 0 & 3 \\ 2 & 3 & 0 & a & 1 \\ 3 & a & 1 & 2 & 0 \\ \hline a & 0 & 3 & 1 & 2 \end{array}\right)$$

$$\theta(g_i), \theta^{-1}(g_j), g_s + \theta(g_s) + \delta \in \{1, 3\}.$$

Hence

$$\theta(g_i) + \theta^{-1}(g_j) \neq g_s + \theta(g_s) + \delta.$$

An example: $Ext(\mathbb{Z}_6; a)$

$$\left(\begin{array}{c|ccccc|c} 0 & 1 & 2 & 3 & 4 & 5 & a \\ \hline 1 & 2 & 3 & a & 5 & 0 & 4 \\ 2 & 3 & 4 & 5 & a & 1 & 0 \\ 3 & 4 & 5 & 0 & 1 & a & 2 \\ 4 & a & 0 & 1 & 2 & 3 & 5 \\ 5 & 0 & a & 2 & 3 & 4 & 1 \\ \hline a & 5 & 1 & 4 & 0 & 2 & 3 \end{array}\right)$$

If this square has an orthogonal mate, then the red cells must be on the same transversal, the blue cells must be on the same transversal, and the green cells must be on the same transversal.

An example: $Ext(\mathbb{Z}_6; a)$

$$\left(\begin{array}{c|ccccc|c} 0 & 1 & 2 & 3 & 4 & 5 & \color{blue}{a} \\ \hline 1 & 2 & 3 & \color{yellow}{a} & 5 & 0 & \color{red}{4} \\ 2 & 3 & 4 & 5 & \color{green}{a} & 1 & \color{cyan}{0} \\ 3 & 4 & 5 & 0 & 1 & \color{cyan}{a} & \color{yellow}{2} \\ 4 & a & 0 & 1 & 2 & 3 & 5 \\ 5 & 0 & a & 2 & 3 & 4 & 1 \\ \hline \color{red}{a} & 5 & 1 & \color{cyan}{4} & \color{yellow}{0} & \color{blue}{2} & \color{green}{3} \end{array}\right)$$

If this square has an orthogonal mate, then the red cells must be on the same transversal, the blue cells must be on the same transversal, and the green cells must be on the same transversal.

Further, the cyan cells must be on the same transversal, and the yellow cells must be on the same transversal.

An example: $Ext(\mathbb{Z}_6; a)$

$$\left(\begin{array}{c|ccccc|c} 0 & 1 & 2 & 3 & 4 & 5 & a \\ \hline 1 & 2 & 3 & a & 5 & 0 & 4 \\ 2 & 3 & 4 & 5 & a & 1 & 0 \\ 3 & 4 & 5 & 0 & 1 & a & 2 \\ 4 & a & 0 & 1 & 2 & 3 & 5 \\ 5 & 0 & a & 2 & 3 & 4 & 1 \\ \hline a & 5 & 1 & 4 & 0 & 2 & 3 \end{array}\right)$$

If this square has an orthogonal mate, then the red cells must be on the same transversal, the blue cells must be on the same transversal, and the green cells must be on the same transversal.

Further, the cyan cells must be on the same transversal, and the yellow cells must be on the same transversal.

We cannot add more transversals: this is a bachelor square.

An example: $Ext(\mathbb{Z}_8; a)$

0	1	2	3	4	5	6	7	a
1	2	3	4	a	6	7	0	5
2	3	4	5	6	a	0	1	7
3	4	5	6	7	0	a	2	1
4	5	6	7	0	1	2	a	3
5	a	7	0	1	2	3	4	6
6	7	a	1	2	3	4	5	0
7	0	1	a	3	4	5	6	2
a	6	0	2	5	7	1	3	4

If this square has an orthogonal mate, then cells of the same color must be on the same transversal.

An example: $Ext(\mathbb{Z}_8; a)$

$$\left(\begin{array}{c|ccccccc|c}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & a \\
\hline
1 & 2 & 3 & 4 & a & 6 & 7 & 0 & 5 \\
2 & 3 & 4 & 5 & 6 & a & 0 & 1 & 7 \\
3 & 4 & 5 & 6 & 7 & 0 & a & 2 & 1 \\
4 & 5 & 6 & 7 & 0 & 1 & 2 & a & 3 \\
5 & a & 7 & 0 & 1 & 2 & 3 & 4 & 6 \\
6 & 7 & a & 1 & 2 & 3 & 4 & 5 & 0 \\
7 & 0 & 1 & a & 3 & 4 & 5 & 6 & 2 \\
\hline
a & 6 & 0 & 2 & 5 & 7 & 1 & 3 & 4
\end{array}\right)$$

If this square has an orthogonal mate, then cells of the same color must be on the same transversal.

Have not determined yet if these partial transversals all complete to transversals.

A generalization.

$$Ext(G; a_1, \ldots, a_n) = \left(\begin{array}{c|ccc|ccc} g_0 & g_1 & \cdots & g_{m-1} & a_1 & \cdots & a_n \\ \hline g_1 & & & & & & \\ \vdots & & A & & & B & \\ g_{m-1} & & & & & & \\ \hline a_1 & & & & & & \\ \vdots & & C & & & D & \\ a_n & & & & & & \end{array}\right)$$

The $g_i g_j$th entry in A is either $g_i + g_j$ or one of $a_1, \ldots, a_n$.

There are several choices for B, C, and D.