The principal Wedderburn Theorem for Jordan superalgebras with unity.

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History

Theodor Mollien (1861 - 1941)

(1892) Let $A$ be a finite-dimensional associative algebra over the complex field, and let $N$ be the solvable radical of $A$. Then there exists a subalgebra $S \subseteq A$ such that $S \cong A/N$ and $A = S \oplus N$.

J. M. Wedderburn (1882 - 1941)

(1905) Let $A$ be a finite-dimensional associative algebra over $F$, and let $N$ be the solvable radical of $A$, then there exists a subalgebra $S \subseteq A$ such that $S \cong A/N$ and $A = S \oplus N$.

- Üeber Systeme Höherer complexer zahlen, Math Ann 41, 1893
History

J. M. Wedderburn (1882 - 1941)

(1905) Let $\mathcal{A}$ be a finite-dimensional associative algebra over $\mathbb{F}$, and let $\mathcal{N}$ be the solvable radical of $\mathcal{A}$, then there exists a subalgebra $S \subseteq \mathcal{A}$ such that $S \cong \mathcal{A}/\mathcal{N}$ and $\mathcal{A} = S \oplus \mathcal{N}$.

History

Adrian A. Albert (1905-1972)

In 1945 proved an analogue to the Principal Wedderburn theorem for finite-dimensional especial Jordan algebras over a field of characteristic zero.

- The Wedderburn principal theorem for Jordan algebras, Ann. of Math. (2)
Penico, Askinuze

Generalized the result of A. Albert for any finite-dimensional Jordan algebra over arbitrary fields with $\text{Char } \mathbb{F} \neq 2$.

- Askinuze, A. Theorem on the splittability of $J$-algebras, Ukrain. Mat. Z. 3 (1951)
Definition

Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be vector spaces over a field $\mathbb{F}$, $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ is called a superalgebra if it is a $\mathbb{Z}_2$-graded algebra over $\mathbb{F}$, it is $\mathcal{A}_i \mathcal{A}_j \subseteq \mathcal{A}_{i+j} \mod 2$.
Definition

Let $A_0$ and $A_1$ be vector spaces over a field $\mathbb{F}$, $A = A_0 \dot{+} A_1$ is called a superalgebra if it is a $\mathbb{Z}_2$-graded algebra over $\mathbb{F}$, it is $A_i A_j \subseteq A_{i+j}$ mod 2.

A superalgebra $A = A_0 \dot{+} A_1$ is called a **Jordan superalgebra** if it satisfies the superidentities

$$a_i a_j = (-1)^{ij} a_j a_i$$

(1)

$$((a_i a_j) a_k) a_l + (-1)^{lk+j+k}((a_i a_l) a_k) a_j + (-1)^{ij+l+k+l}((a_j a_l) a_k) a_i = (a_i a_j)(a_k a_l) + (-1)^{lk+j} (a_l a_i) (a_j a_k) + (-1)^{kj} (a_i a_k) (a_j a_l)$$

(2)

for all $a_i, a_j, a, a_k \in \tilde{J}_0 \cup \tilde{J}_1$. 
Definition

Let \( A_0 \) and \( A_1 \) be vector spaces over a field \( \mathbb{F} \), \( A = A_0 \oplus A_1 \) is called a superalgebra if it is a \( \mathbb{Z}_2 \)-graded algebra over \( \mathbb{F} \), it is \( A_iA_j \subseteq A_{i+j} \mod 2 \)

A superalgebra \( A = A_0 \oplus A_1 \) is called a **Jordan superalgebra** if it satisfies the superidentities

\[
a_ia_j = (-1)^{ij} a_ja_i
\]  
(1)

\[
((a_ia_j)a_k)a_l + (-1)^{lk+l'j+k}((a_ia_l)a_k)a_j + (-1)^{ij+l'k+l+l'k}((a_ja_l)a_k)a_i =
\]

\[
(a_ia_j)(a_k a_l) + (-1)^{lk+l'j}(a_ia_l)(a_ja_k) + (-1)^{kj}(a_ia_k)(a_ja_l)
\]  
(2)

for all \( a_i, a_j, a, a_k \in J_0 \cup J_1 \).

Definition

An \( A \)-superbimodule \( M = M_0 \oplus M_1 \) is called a **Jordan superbimodule** if the corresponding split null extension superalgebra \( E = A \oplus M \) is Jordan superalgebra.
Some Examples

Let $\mathcal{A}$ be an associative superalgebra and consider the new multiplication in $\mathcal{A}$,

$$x \circ y = \frac{1}{2}(xy + (-1)^{|x||y|}yx)$$

The new superalgebra is a Jordan superalgebra and we denoted this by $\mathcal{A}^+$. 

Let $A$ be an associative superalgebra and consider the new multiplication in $A$,

$$x \circ y = \frac{1}{2}(xy + (-1)^{|x||y|}yx)$$

The new superalgebra is a Jordan superalgebra and we denoted this by $A^+$. In particular, we can consider the associative superalgebra

$$M_{n+m}(\mathbb{F}) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} + \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$$

and denote this Jordan superalgebra by $M_{n|m}(\mathbb{F})^+$. 

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The PWT for Jordan superalgebras with 1.
Let $V = V_0 \oplus V_1$ be a graded $\mathbb{F}$-vectorspace and let $f : V \times V \rightarrow \mathbb{F}$ be a superform, i.e $f|_{V_0}$, $(f|_{V_1})$ is a symmetric form (skew form) and $f(V_0, V_1) = 0$. It is easy to check that $\mathcal{J} = \mathbb{F} \cdot 1 + V_0 \cdot V_1$ with multiplication $v \cdot 1 = v$, $v \cdot w = f(v, w)$ is a Jordan superalgebra. It is called the Jordan superalgebra of superform.
Let \( V = V_0 \oplus V_1 \) be a graded \( \mathbb{F} \)-vectorspace and let \( f : V \times V \rightarrow \mathbb{F} \) be a superform, i.e \( f|_{V_0}, (f|_{V_1}) \) is a symmetric form (skew form) and \( f(V_0, V_1) = 0 \). It is easy to check that \( \mathcal{J} = \mathbb{F} \cdot 1 + V_0 \perp V_1 \) with multiplication \( v \cdot 1 = v, v \cdot w = f(v, w) \) is a Jordan superalgebra. It is called the Jordan superalgebra of superform.

Let \( t \in \mathbb{F} \) and \( \mathcal{D}_t = (\mathbb{F} \cdot e_1 + \mathbb{F} \cdot e_2) \perp (\mathbb{F} \cdot x + \mathbb{F} \cdot y) \) be a parametric family of superalgebras with multiplication

\[
e_i \cdot x = \frac{1}{2}x, \quad e_i \cdot y = \frac{1}{2}y, \quad x \cdot y = -y \cdot x = e_1 + te_2
\]

This superalgebra is a Jordan superalgebra.
Let $V = V_0 \oplus V_1$ be a graded $F$-vectorspace and let $f : V \times V \rightarrow F$ be a superform, i.e. $f|_{V_0}$, $(f|_{V_1})$ is a symmetric form (skew form) and $f(V_0, V_1) = 0$. It is easy to check that $\mathcal{J} = F \cdot 1 + V_0 \hat{\dagger} V_1$ with multiplication $v \cdot 1 = v, v \cdot w = f(v, w)$ is a Jordan superalgebra. It is called the Jordan superalgebra of superform.

Let $t \in F$ and $D_t = (F \cdot e_1 + F \cdot e_2) \hat{\dagger} (F \cdot x + F \cdot y)$ be a parametric family of superalgebras with multiplication
\[
e_i \cdot x = \frac{1}{2}x, \quad e_i \cdot y = \frac{1}{2}y, = x \cdot y = -y \cdot x = e_1 + te_2
\]

This superalgebra is a Jordan superalgebra.

Let $\mathcal{K}_3 = F \cdot e_1 \hat{\dagger} (F \cdot x + F \cdot y)$ be a superalgebra with multiplication
\[
e_1 \cdot x = \frac{1}{2}x, \quad e_1 \cdot y = \frac{1}{2}y, = x \cdot y = -y \cdot x = e_1
\]

This superalgebra is a Jordan superalgebra. It is called the Kaplansky superalgebra.
V. Kac introduced the Jordan superalgebra of dimension 10, $K_{10}$.

Over a field $F$, $\text{Char} F = 0$, any Jordan superalgebra of the list above is a simple Jordan superalgebra.
Let $A$ be a Jordan superalgebra and let $N$ be the solvable radical of $A$. When there exists a subsuperalgebra $S \subseteq A$ such that $S \cong A/N$ and $A = S \oplus N$?
Let $\mathcal{A}$ be a Jordan superalgebra and let $\mathcal{N}$ be the solvable radical of $\mathcal{A}$. When there exists a subsuperalgebra $S \subseteq \mathcal{A}$ such that $S \cong \mathcal{A}/\mathcal{N}$ and $\mathcal{A} = S \oplus \mathcal{N}$?

This problem is an analogue to the validity of the Principal Wedderburn Theorem (PWT) for associative algebras.
Proposition

If the principal Wedderburn theorem is valid for Jordan superalgebras with unity, then it is valid for any Jordan superalgebra.
Reduction preliminaries

**Proposition**

If the principal Wedderburn theorem is valid for Jordan superalgebras with unity, then it is valid for any Jordan superalgebra.

**Proposition**

Let $\mathcal{J}$ be a finite dimensional semisimple Jordan superalgebra, that is, $\mathcal{N}(\mathcal{J}) = 0$ where $\mathcal{N}$ is the soluble radical. Fix a class $\mathcal{M}(\mathcal{J})$ of finite dimensional Jordan $\mathcal{J}$-bimodules which is closed with respect to subbimodules and homomorphic images. Denote by $\mathcal{K}_\mathcal{M}(\mathcal{J})$ the class of finite dimensional Jordan superalgebras $A$ that satisfy the following conditions: $A/\mathcal{N}(A) \cong \mathcal{J}$, $\mathcal{N}(A)^2 = 0$ and, $\mathcal{N}(A)$ considered as $\mathcal{J}$-bimodule belongs to $\mathcal{M}(\mathcal{J})$. Then if PWT is true for all superalgebras $B \in \mathcal{K}_\mathcal{M}(\mathcal{J})$ with $\mathcal{N}(B)$ an irreducible $\mathcal{J}$-bimodule, then it is true for all superalgebras $A$ from $\mathcal{K}_\mathcal{M}(\mathcal{J})$. 
irreducible bimodules over Jordan superalgebras

Irreducible bimodules over Jordan superalgebras of type $\mathcal{M}_{n|m}(\mathbb{F})^{(+)}$, $\mathcal{D}_t$, Kaplansky, and superform were classified by Zelmanov-Martinez. The cases of Jordan superalgebras of type $K_{10}$ was proved by Shtern.
As a first step we consider the case in which the radical satisfies \( N^2 = 0 \), and the quotient superalgebra \( \mathfrak{j}/\mathcal{N} \) is a simple Jordan superalgebra of one of the following types: \( M_{n|m}(\mathbb{F})(^+) \), superforms, \( D_t \), or \( K_{10} \), we prove that an analogue to the PWT is valid, provided some restrictions are imposed on the types of irreducible bimodules contained in the radical \( \mathcal{N} \). The restrictions are necessary and counter-examples were provided.
Main Theorem

Theorem (Main Theorem)

Let $\mathcal{J}, \mathcal{N}$ be as before. In the following cases there exists a subsuperalgebra $S \subseteq \mathcal{J}$ such that $S \cong \mathcal{J}/\mathcal{N}$ and $\mathcal{J} = S \oplus \mathcal{N}$:

$\mathcal{J}/\mathcal{N}$ is isomorphic to $M_{n|m}(F)^{(+)}$. And when $n + m \geq 3$, $\mathcal{N}$ does not contain any copy of the regular bimodule $\text{Reg} M_{n|m}(F)^{(+)}$. When $m = n = 1$, $\mathcal{N}$ does not contain any copy of the regular bimodule nor of $V_{e}$. $\mathcal{J}/\mathcal{N}$ is a superalgebra of a superform with even part of dimension $n$, and $\mathcal{N}$ does not contain any copy of the irreducible bimodule $C_{n}/C_{n-2}$ when $n$ is odd, or of $u \cdot C_{n}/u \cdot C_{n-2}$ when $n$ is even. $\mathcal{J}/\mathcal{N}$ is isomorphic to $D_{t}$, $t \neq -1$. And $\mathcal{N}$ does not contain any copy of the bimodule $\text{Reg} D_{t}$, or of the vector space generated by one even vector. $\mathcal{J}/\mathcal{N}$ is a Kac superalgebra without restriction in the bimodule.
Main Theorem

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Let $\mathcal{J}, \mathcal{N}$ be as before. In the following cases there exists a subsuperalgebra $S \subseteq \mathcal{J}$ such that $S \cong \mathcal{J}/\mathcal{N}$ and $\mathcal{J} = S \oplus \mathcal{N}$:

- $\mathcal{J}/\mathcal{N}$ is isomorphic to $\mathcal{M}_{n|m}(\mathbb{F})(^+)$. And when $n + m \geq 3$, $\mathcal{N}$ does not contain any copy of the regular bimodule $\text{Reg} \mathcal{M}_{n|m}(\mathbb{F})(^+)$. When $m = n = 1$ $\mathcal{N}$ does not contain any copy of the regular bimodule nor of $V^e$. 
Let $\mathcal{J}, \mathcal{N}$ be as before. In the following cases there exists a subsuperalgebra $S \subseteq \mathcal{J}$ such that $S \cong \mathcal{J}/\mathcal{N}$ and $\mathcal{J} = S \oplus \mathcal{N}$:

- $\mathcal{J}/\mathcal{N}$ is isomorphic to $\mathcal{M}_{n|m}(F)^{(+)}$. And when $n + m \geq 3$, $\mathcal{N}$ does not contain any copy of the regular bimodule $\text{Reg} \mathcal{M}_{n|m}(F)^{(+)}$. When $m = n = 1$ $\mathcal{N}$ does not contain any copy of the regular bimodule nor of $\mathcal{V}^e$.

- $\mathcal{J}/\mathcal{N}$ is a superalgebra of a superform with even part of dimension $n$, and $\mathcal{N}$ does not contain any copy of the irreducible bimodule $C_n/C_{n-2}$ when $n$ is odd, or of $u \cdot C_n/u \cdot C_{n-2}$ when $n$ is even.
Main Theorem

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Let $\mathcal{J}, \mathcal{N}$ be as before. In the following cases there exists a subsuperalgebra $S \subseteq \mathcal{J}$ such that $S \cong \mathcal{J}/\mathcal{N}$ and $\mathcal{J} = S \oplus \mathcal{N}$:

- $\mathcal{J}/\mathcal{N}$ is isomorphic to $\mathcal{M}_{n|m}(\mathbb{F})(^+)$. And when $n + m \geq 3$, $\mathcal{N}$ does not contain any copy of the regular bimodule $\text{Reg} \mathcal{M}_{n|m}(\mathbb{F})(^+)$. When $m = n = 1$ $\mathcal{N}$ does not contain any copy of the regular bimodule nor of $V^e$.

- $\mathcal{J}/\mathcal{N}$ is a superalgebra of a superform with even part of dimension $n$, and $\mathcal{N}$ does not contain any copy of the irreducible bimodule $\mathcal{C}_n/\mathcal{C}_{n-2}$ when $n$ is odd, or of $u \cdot \mathcal{C}_n/\mathcal{C}_{n-2}$ when $n$ is even.

- $\mathcal{J}/\mathcal{N}$ is isomorphic to $D_t$, $t \neq -1$. And $\mathcal{N}$ does not contain any copy of the bimodule $\text{Reg} D_t$, or of the vector space generated by one even vector.
Main Theorem

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Let $\mathcal{J}, \mathcal{N}$ be as before. In the following cases there exists a subsuperalgebra $S \subseteq \mathcal{J}$ such that $S \cong \mathcal{J}/\mathcal{N}$ and $\mathcal{J} = S \oplus \mathcal{N}$:

- $\mathcal{J}/\mathcal{N}$ is isomorphic to $\mathcal{M}_{n|m}(\mathbb{F})^{(+)}$. And when $n + m \geq 3$, $\mathcal{N}$ does not contain any copy of the regular bimodule $\text{Reg} \mathcal{M}_{n|m}(\mathbb{F})^{(+)}$. When $m = n = 1$ $\mathcal{N}$ does not contain any copy of the regular bimodule nor of $V^e$.
- $\mathcal{J}/\mathcal{N}$ is a superalgebra of a superform with even part of dimension $n$, and $\mathcal{N}$ does not contain any copy of the irreducible bimodule $\mathcal{C}_n/\mathcal{C}_{n-2}$ when $n$ is odd, or of $u \cdot \mathcal{C}_n/u \cdot \mathcal{C}_{n-2}$ when $n$ is even.
- $\mathcal{J}/\mathcal{N}$ is isomorphic to $\mathcal{D}_t$, $t \neq -1$. And $\mathcal{N}$ does not contain any copy of the bimodule $\text{Reg} \mathcal{D}_t$, or of the vector space generated by one even vector.
- $\mathcal{J}/\mathcal{N}$ is a Kac superalgebra without restriction in the bimodule.
The restrictions imposed by the theorem above are essential, and we provide the corresponding counter-examples.
Irreducible bimodules over Jordan superalgebras of superform

Let $V = V_0 + V_1$ be a vector superspace equipped with a nondegenerate superform, and suppose that $V_1 \neq 0$. Let $v_1, \ldots, v_n$ be an orthogonal basis of $V_0$ and $w_1, \ldots, w_{2m}$ be a basis of $V_1$ such that $(w_{2i-1}, w_{2i}) = 1$, $1 \leq i \leq m$, where all other products are zero.

Let $C$ be the Clifford algebra over $\mathbb{F}$. Let $0 \leq i_1, \ldots, i_n \leq 1$ and $k_1, \ldots, k_{2m}$ are non negative integers, the elements $v_1^{i_1} \cdots v_n^{i_n} w_1^{k_1} \cdots w_{2m}^{k_{2m}}$, form a basis for $C$.

Consider the subspace $C_r = \sum_{i \leq r} V \cdots V$ as the span of all basic products of length $\leq r$.

$$\mathbb{F} = C_0 \subseteq C_1 \subseteq \cdots ; \quad C = \bigcup_{r \geq 0} C_r$$
Irreducible bimodules over a Jordan superalgebras of superform

Any superspace of type $C_r$ with $r$ odd integer is a superbimodule over the Jordan superalgebra of superform.

Theorem (C. Martinez- E. Zelmanov)

The only finite dimensional unital irreducible Jordan bimodules over $J = F \cdot 1 + V$, (Jordan superalgebra of superform) are $C_r/C_r-2$ if $r$ is odd and $uC_r/uC_r-2$ if $r$ is even, where $u$ is an even vector.
Any superspace of type $C_r$ with $r$ odd integer is a superbimodule over the Jordan superalgebra of superform.

**Theorem (C. Martinez- E. Zelmanov)**

The only finite dimensional unital irreducible Jordan bimodules over $\mathfrak{J} = \mathbb{F} \cdot 1 + V$, (Jordan superalgebra of superform) are $C_r/C_{r-2}$ if $r$ is odd and $uC_r/uC_{r-2}$ if $r$ is even, where $u$ is an even vector.
A counter-example to the PWT in Jordan superalgebras of superform

We consider the superspace \( N \subset A \) where

\[
A_0 = \text{Vect} \langle 1, v_i, v_1 v_2, v_1 v_3, v_2 v_3, v_1 v_2 v_3, v_i w_s^2, v_i w_1 w_2, w_s^2, w_1 w_2 \rangle
\]

for \( i=1,2,3, \ s=1,2 \)
A counter-example to the PWT in Jordan superalgebras of superform

We consider the superspace $\mathcal{N} \subset A$ where

$A_0 = \text{Vect} \langle 1, v_i, v_1 v_2, v_1 v_3, v_2 v_3, v_i w_s^2, v_i w_1 w_2, w_s^2, w_1 w_2 \rangle$

$A_1 = \text{Vect} \langle w_1, w_2, v_i w_s, v_1 v_2 w_s, v_1 v_3 w_s, v_2 v_3 w_s, w_s^3, w_s^2 w_t \rangle$
A counter-example to the PWT in Jordan superalgebras of superform

We consider the superspace $\mathcal{N} \subset \mathcal{A}$ where

$$\mathcal{A}_0 = \text{Vect} \langle 1, v_i, v_1 v_2, v_1 v_3, v_2 v_3, v_1 v_2 v_3, v_i w_2^2, v_i w_1 w_2, w_2^2, w_1 w_2 \rangle$$

$$\mathcal{N}_0 = \text{Vect} \langle v_1 v_2, v_1 v_3, v_2 v_3, v_1 v_2 v_3, v_i w_2^2, v_i w_1 w_2, w_2^2, w_1 w_2 \rangle$$
A counter-example to the PWT in Jordan superalgebras of superform

We consider the superspace $\mathcal{N} \subset \mathcal{A}$ where

$$\mathcal{A}_1 = \text{Vect} \langle w_1, w_2, v_i w_s, v_1 v_2 w_s, v_1 v_3 w_s, v_2 v_3 w_s, w_s^3, w_s^2 w_t \rangle$$

$$\mathcal{N}_1 = \text{Vect} \langle v_i w_s, v_1 v_2 w_s, v_1 v_3 w_s, v_2 v_3 w_s, w_s^3, w_s^2 w_t \rangle$$
A counter-example to the PWT in Jordan superalgebras of superform

We consider the superspace $\mathcal{N} \subset \mathcal{A}$ where

\begin{align*}
\mathcal{A}_0 &= \text{Vect} \langle 1, v_i, v_1 v_2, v_1 v_3, v_2 v_3, v_1 v_2 v_3, v_i w_s^2, v_i w_1 w_2, w_s^2, w_1 w_2 \rangle \\
\mathcal{A}_1 &= \text{Vect} \langle w_1, w_2, v_i w_s, v_1 v_2 w_s, v_1 v_3 w_s, v_2 v_3 w_s, w_s^3, w_s^2 w_t \rangle \\
\mathcal{N}_0 &= \text{Vect} \langle v_1 v_2, v_1 v_3, v_2 v_3, v_1 v_2 v_3, v_i w_s^2, v_i w_1 w_2, w_s^2, w_1 w_2 \rangle \\
\mathcal{N}_1 &= \text{Vect} \langle v_i w_s, v_1 v_2 w_s, v_1 v_3 w_s, v_2 v_3 w_s, w_s^3, w_s^2 w_t \rangle
\end{align*}

and we observe that

\[
\mathcal{A}/\mathcal{N} \cong (F \cdot 1 + F \cdot v_1 + F \cdot v_2 + F \cdot v_3) \oplus (F \cdot w_1 + F \cdot w_2)
\]
A counter-example

We define the nonzero products on $A$, as follows $v_i^2 = 1$ for $i = 1, 2, 3$ and

\[ w_1 \cdot w_2 = 1 + v_1 v_2 v_3 \]

\[ v_i \circ v_j = \frac{1}{2} (v_i v_j + v_j v_i) \]

\[ 0 = v_i \circ w_s = v_i w_s + w_s v_i \]
A counter-example

We define the nonzero products on $A$, as follows $v_i^2 = 1$ for $i = 1, 2, 3$ and

$w_1 \cdot w_2 = 1 + v_1 v_2 v_3$

$v_1 \cdot v_2 v_3 = v_1 v_2 v_3, \quad v_2 \cdot v_1 v_3 = -v_1 v_2 v_3, \quad v_3 \cdot v_1 v_2 = v_1 v_2 v_3,$
A counter-example

We define the nonzero products on $\mathcal{A}$, as follows $v_i^2 = 1$ for $i = 1, 2, 3$ and $w_1 \cdot w_2 = 1 + v_1 v_2 v_3$

\[
\begin{align*}
 v_1 \cdot v_2 v_3 &= v_1 v_2 v_3, & v_2 \cdot v_1 v_3 &= -v_1 v_2 v_3, & v_3 \cdot v_1 v_2 &= v_1 v_2 v_3, \\
v_i \cdot w_1 w_2 &= v_i w_1 w_2, & v_i \cdot w_s^2 &= v_i w_s^2, & v_i \cdot v_i w_s^2 &= w_s^2, \\
v_i \cdot v_i w_1 w_2 &= w_1 w_2, & v_1 \cdot v_2 w_s &= v_1 v_2 w_s, & v_1 \cdot v_3 w_s &= v_1 v_3 w_s, \\
v_2 \cdot v_3 w_s &= v_2 v_3 w_s, & v_i \cdot v_i v_j w_s &= v_j w_s, & v_2 \cdot v_3 w_s &= v_2 v_3 w_s, \\
\end{align*}
\]
We define the nonzero products on $A$, as follows $v_i^2 = 1$ for $i = 1, 2, 3$ and $w_1 \cdot w_2 = 1 + v_1 v_2 v_3$

\begin{align*}
  v_1 \cdot v_2 v_3 &= v_1 v_2 v_3, \\
  v_i \cdot w_1 w_2 &= v_i w_1 w_2, \\
  v_i \cdot v_i w_1 w_2 &= w_1 w_2, \\
  v_2 \cdot v_3 w_s &= v_2 v_3 w_s, \\
  w_s \cdot v_i v_j &= v_i v_j w_s, \\
  v_2 \cdot v_1 v_3 &= -v_1 v_2 v_3, \\
  v_i \cdot w_s^2 &= v_i w_s^2, \\
  v_1 \cdot v_2 w_s &= v_1 v_2 w_s, \\
  v_i \cdot v_i v_j w_s &= v_j w_s, \\
  w_1 \cdot v_i v_j w_2 &= v_i v_j, \\
  w_2 \cdot v_i v_j w_1 &= -v_i v_j, \\
  v_3 \cdot v_1 v_2 &= v_1 v_2 v_3, \\
  v_i \cdot v_i w_s^2 &= w_s^2, \\
  v_1 \cdot v_3 w_s &= v_1 v_3 w_s.
\end{align*}
A counter-example

We define the nonzero products on $A$, as follows $v_i^2 = 1$ for $i = 1, 2, 3$ and $w_1 \cdot w_2 = 1 + v_1 v_2 v_3$

\[
\begin{align*}
v_1 \cdot v_2 v_3 &= v_1 v_2 v_3, \\
v_i \cdot w_1 w_2 &= v_i w_1 w_2, \\
v_i \cdot v_i w_1 w_2 &= w_1 w_2, \\
v_2 \cdot v_3 w_s &= v_2 v_3 w_s, \\
w_s \cdot v_i v_j &= v_i v_j w_s, \\
w_1 \cdot w_2^3 &= 3w_2^2, \\
w_2 \cdot w_3^3 &= -3w_1^2, \\
w_1 \cdot v_1 w_2^2 &= -2v_1 w_2, \\
w_2 \cdot v_1 w_2^2 &= -2v_1 w_1, \\
v_2 \cdot v_1 v_3 &= -v_1 v_2 v_3, \\
v_i \cdot w_s^2 &= v_i w_s^2, \\
v_2 \cdot v_1 v_3 &= -v_1 v_2 v_3, \\
v_3 \cdot v_1 v_2 &= v_1 v_2 v_3, \\
v_i \cdot v_i w_s^2 &= w_s^2, \\
v_1 \cdot v_2 w_s &= v_1 v_2 w_s, \\
v_1 \cdot v_3 w_s &= v_1 v_3 w_s, \\
v_1 \cdot v_1 v_j w_s &= v_j w_s, \\
w_1 \cdot v_i v_j w_2 &= v_i v_j, \\
w_1 \cdot w_1 w_2^2 &= 2w_1 w_2, \\
w_2 \cdot w_1 w_2^2 &= -w_2^2, \\
w_2 \cdot w_1 w_2 &= 2w_1 w_2 \\
w_2 \cdot v_1 w_2 &= v_1 w_2, \\
w_2 \cdot v_1 w_1 &= -v_1 w_1, \\
w_2 \cdot v_1 v_1 &= -v_1 w_1, \\
w_2 \cdot v_1 w_1 &= v_1 w_2, \\
w_2 \cdot w_1 w_2 &= 2w_1 w_2 \\
w_2 \cdot v_i w_s &= v_i w_s + w_s v_i \\
w_1 \cdot w_2 &= 1 + v_1 v_2 v_3 \\
w_1 \cdot w_1 &= 1 + v_1 v_2 v_3 \\
1 &= w_1 \circ w_2 = \frac{1}{2}(w_1 w_2 - w_2 w_1)
\end{align*}
\]
A counter-example

It is easy to check that the superspace $\mathcal{A}$ with the multiplication above is a Jordan superalgebra and the quotient superalgebra is isomorphic to superalgebra of superform

$$\mathcal{A}/\mathcal{N} \cong (F \cdot 1 + F \cdot v_1 + F \cdot v_2 + F \cdot v_3) + (F \cdot w_1 + F \cdot w_2) \text{ e } \mathcal{N} \cong C_3/C_1.$$
A counter-example

it is easy to check that the superspace $\mathcal{A}$ with the multiplication above is a Jordan superalgebra and the quotient superalgebra is isomorphic to superalgebra of superform

$$\mathcal{A}/\mathcal{N} \cong (\mathbb{F} \cdot 1 + \mathbb{F} \cdot v_1 + \mathbb{F} \cdot v_2 + \mathbb{F} \cdot v_3) + (\mathbb{F} \cdot w_1 + \mathbb{F} \cdot w_2) \quad \text{and} \quad \mathcal{N} \cong C_3/C_1.$$ 

In particular, we have that

$$\mathcal{A}_0/\mathcal{N}_0 \cong \mathfrak{J}(V_0, f) = \mathbb{F} \cdot 1 + \mathbb{F} \cdot v_1 + \mathbb{F} \cdot v_2 + \mathbb{F} \cdot v_3$$

is a Jordan algebra.
A counter-example

it is easy to check that the superspace $\mathcal{A}$ with the multiplication above is a Jordan superalgebra and the quotient superalgebra is isomorphic to superalgebra of superform

$$\mathcal{A}/\mathcal{N} \cong (F \cdot 1 + F \cdot \nu_1 + F \cdot \nu_2 + F \cdot \nu_3) + (F \cdot w_1 + F \cdot w_2) \oplus \mathcal{N} \cong C_3/C_1.$$  

In particular, we have that

$$\mathcal{A}_0/\mathcal{N}_0 \cong \mathcal{J}(V_0, f) = F \cdot 1 + F \cdot \nu_1 + F \cdot \nu_2 + F \cdot \nu_3$$

is a Jordan algebra.

Since the PWT is valid for Jordan algebras, then there exist $\tilde{\nu}_i \in \mathcal{A}_0$ such that $\tilde{\nu}_i \equiv \nu_i \mod \mathcal{N}_0$ and $\tilde{\nu}_i^2 = 1$, $\tilde{\nu}_i \cdot \tilde{\nu}_j = 0 \; i \neq j$. 
A counter-example

it is easy to check that the superspace $\mathcal{A}$ with the multiplication above is a Jordan superalgebra and the quotient superalgebra is isomorphic to superalgebra of superform

$$\mathcal{A}/\mathcal{N} \cong (F \cdot 1 + F \cdot \nu_1 + F \cdot \nu_2 + F \cdot \nu_3) \oplus (F \cdot w_1 + F \cdot w_2) \text{ e } \mathcal{N} \cong C_3/C_1.$$ 

In particular, we have that

$$\mathcal{A}_0/\mathcal{N}_0 \cong \tilde{J}(V_0, f) = F \cdot 1 + F \cdot \nu_1 + F \cdot \nu_2 + F \cdot \nu_3$$

is a Jordan algebra.

Since the PWT is valid for Jordan algebras, then there exist $\tilde{\nu}_i \in \mathcal{A}_0$ such that $\tilde{\nu}_i \equiv \nu_i \mod \mathcal{N}_0$ and $\tilde{\nu}_i^2 = 1$, $\tilde{\nu}_i \cdot \tilde{\nu}_j = 0$ $i \neq j$. We assume $\tilde{\nu}_i = \nu_i$. 

The PWT for Jordan superalgebras with 1.
A counter-example

\[(A/\mathcal{N})_1 \cong F \cdot w_1 + F \cdot w_2\]

If the PWT is valid for \(A\) then there exists elements \(\tilde{w}_1, \tilde{w}_2 \in A_1\) such that \(\tilde{w}_s \equiv w_s\) and \(\tilde{w}_1 \cdot \tilde{w}_2 = 1, \ \tilde{w}_s \cdot v_i = 0\).
We recall that
\[ \mathcal{N}_1 = \text{Vect} \langle v_i w_s, v_1 v_2 w_s, v_1 v_3 w_s, v_2 v_3 w_s, w_s^3, w_s^2 w_t \rangle, \] and we can adopt
A counter-example

We recall that
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and we can adopt
\[ \tilde{w}_i = w_i + \sum_{\omega \in \mathcal{N}_1} \alpha_{\omega} \omega, \]
A counter-example

We recall that
\[ \mathcal{N}_1 = \text{Vect} \left\langle v_i w_s, v_1 v_2 w_s, v_1 v_3 w_s, v_2 v_3 w_s, w_s^3, w_s^2 w_t \right\rangle, \]
and we can adopt
\[ \tilde{w}_i = w_i + \sum_{\omega \in \mathcal{N}_1} \alpha_{\omega} \omega, \]
Now, it is possible to prove that
\[
\begin{align*}
\tilde{w}_1 &= w_1 + \alpha_{11} w_1^3 + \alpha_{21} w_1^2 w_2 + \alpha_{31} w_1 w_2^2 + \alpha_{41} w_2^3 \\
\tilde{w}_2 &= w_2 + \alpha_{12} w_1^3 + \alpha_{22} w_1^2 w_2 + \alpha_{32} w_1 w_2^2 + \alpha_{42} w_2^3
\end{align*}
\]
, but \( v_1 v_2 v_3 \) is a non zero vector.
A counter-example

We recall that
\[ \mathcal{N}_1 = \text{Vect} \langle v_i w_s, v_1 v_2 w_s, v_1 v_3 w_s, v_2 v_3 w_s, w_3^3, w_2^2 w_t \rangle, \]
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\end{align*}
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therefore \( \tilde{w}_1 \cdot \tilde{w}_2 = 1 \)

, but \( v_1 v_2 v_3 \) is a non zero vector.
A counter-example

We recall that \( w_1 \cdot w_2 = 1 + v_1 v_2 v_3 \)

\[
\tilde{w}_1 = w_1 + \alpha_{11} w_1^3 + \alpha_{21} w_1^2 w_2 + \alpha_{31} w_1 w_2^2 + \alpha_{41} w_2^3
\]
\[
\tilde{w}_2 = w_2 + \alpha_{12} w_1^3 + \alpha_{22} w_1^2 w_2 + \alpha_{32} w_1 w_2^2 + \alpha_{42} w_2^3
\]

therefore \( \tilde{w}_1 \cdot \tilde{w}_2 = 1 \) if and only if \( v_1 v_2 v_3 = 0 \), but \( v_1 v_2 v_3 \) is a non zero vector.
References

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THANKS FOR YOUR ATTENTION!