The principal Wedderburn Theorem for Jordan superalgebras with unity.

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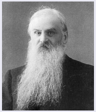
Third Mile High Conference, Denver 2013

Theodor Mollien (1861 - 1941)



(1892) Let $\mathcal{A}$ be a finite-dimensional associative algebra over the complex field, and let $\mathcal{N}$ be the solvable radical of $\mathcal{A}$. Then there exists a subalgebra $\mathcal{S} \subseteq \mathcal{A}$ such that $\mathcal{S} \cong \mathcal{A}/\mathcal{N}$ and $\mathcal{A} = \mathcal{S} \oplus \mathcal{N}$.

- Üeber Systeme Höherer complexer zahlen, Math Ann 41, 1893

History

J. M. Wedderburn (1882 - 1941)

(1905) Let $\mathcal{A}$ be a finite-dimensional associative algebra over $\mathbb{F}$, and let $\mathcal{N}$ be the solvable radical of $\mathcal{A}$, then there exists a subalgebra $\mathcal{S} \subseteq \mathcal{A}$ such that $\mathcal{S} \cong \mathcal{A}/\mathcal{N}$ and $\mathcal{A} = \mathcal{S} \oplus \mathcal{N}$.

- On the structure of hypercomplex number systems, Amer. Math. Soc. Volume 12, Number 2 (1905).

History

Adrian A. Albert (1905-1972)

In 1945 proved an analogue to the Principal Wedderburn theorem for finite-dimensional especial Jordan algebras over a field of characteristic zero.

- The Wedderburn principal theorem for Jordan algebras, Ann. of Math. (2)

Penico, Askinuze

Generalized the result of A. Albert for any finite-dimensional Jordan algebra over arbitrary fields with Char $\mathbb{F} \neq 2$.

- Askinuze, A Theorem on the splittability of J-algebras, Ukrain.Mat.Z. **3** (1951)
- Penico, A.J. The Wedderburn principal theorem for Jordan algebras, Trans. Amer. Math. Soc. **70** (1951),

Definition

Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be vector spaces over a field $\mathbb{F}$, $\mathcal{A} = \mathcal{A}_0 \dot{+} \mathcal{A}_1$ is called a superalgebra if it is a $\mathbb{Z}_2$-graded algebra over $\mathbb{F}$, it is $\mathcal{A}_i\mathcal{A}_j \subseteq \mathcal{A}_{i+j \mod 2}$

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A superalgebra $\mathcal{A} = \mathcal{A}_0 \dot{+} \mathcal{A}_1$ is called a Jordan superalgebra if it satisfies the superidentities

$$a_i a_j = (-1)^{ij} a_j a_i \tag{1}$$

$$((a_i a_j)a_k)a_l + (-1)^{lk+lj+kj}((a_i a_l)a_k)a_j + (-1)^{ij+ik+il+lk}((a_j a_l)a_k)a_i = (a_i a_j)(a_k a_l) + (-1)^{lk+lj}(a_i a_l)(a_j a_k) + (-1)^{kj}(a_i a_k)(a_j a_l) \tag{2}$$

for all $a_i, a_j, a_, a_k \in \mathfrak{J}_0 \cup \mathfrak{J}_1$.

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Definition

An $\mathcal{A}$-superbimodule $\mathcal{M} = \mathcal{M}_0 \dot{+} \mathcal{M}_1$ is called a Jordan superbimodule if the corresponding split null extension superalgebra $\mathcal{E} = \mathcal{A} \oplus \mathcal{M}$ is Jordan superalgebra.

Some Examples

- Let $\mathcal{A}$ be an associative superalgebra and consider the new multiplication in $\mathcal{A}$,

$$x \circ y = \frac{1}{2}(xy + (-1)^{|x||y|}yx)$$

The new superalgebra is a Jordan superalgebra and we denoted this by $\mathcal{A}^+$.

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In particular, we can consider the associative superalgebra

$$\mathcal{M}_{n+m}(\mathbb{F}) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \dot{+} \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$$

and denote this Jordan superalgebra by $\mathcal{M}_{n|m}(\mathbb{F})^{(+)}$.

- Let $V = V_0 \oplus V_1$ be a graded $\mathbb{F}$-vectorspace and let $f : V \times V \longrightarrow \mathbb{F}$ be a superform, i.e $f|_{V_0}$, $(f|_{V_1})$ is a symmetric form (skew form) and $f(V_0, V_1) = 0$. Is easy to check that $\mathfrak{J} = \mathbb{F} \cdot 1 + V_0 \dot{+} V_1$ with multiplication $v \cdot 1 = v$, $v \cdot w = f(v, w)$ is a Jordan superalgebra. It is called the Jordan superalgebra of superform.

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- Let $t \in \mathbb{F}$ and $\mathcal{D}_t = (\mathbb{F} \cdot e_1 + \mathbb{F} \cdot e_2) \dot{+} (\mathbb{F} \cdot x + \mathbb{F} \cdot y)$ be a parametric family of superalgebras with multiplication

$$e_i \cdot x = \frac{1}{2}x,\ e_i \cdot y = \frac{1}{2}y, =\ x \cdot y = -y \cdot x = e_1 + te_2$$

This superalgebra is a Jordan superalgebra.

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  This superalgebra is a Jordan superalgebra.
- Let $\mathcal{K}_3 = \mathbb{F} \cdot e_1 \dot{+} (\mathbb{F} \cdot x + \mathbb{F} \cdot y)$ be a superalgebra with multiplication

$$e_1 \cdot x = \frac{1}{2}x,\ e_1 \cdot y = \frac{1}{2}y, = \ x \cdot y = -y \cdot x = e_1$$

  This superalgebra is a Jordan superalgebra. It is called the Kaplansky superalgebra.

- V.Kac introduced the Jordan superalgebra of dimension 10, K_{10}.

Over a field $\mathbb{F}$, Char$\mathbb{F} = 0$, any Jordan superalgebra of the list above is a simple Jordan superalgebra

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Let $\mathcal{A}$ be a Jordan superalgebra and let $\mathcal{N}$ be the solvable radical of $\mathcal{A}$. When there exists a subsuperalgebra $\mathcal{S} \subseteq \mathcal{A}$ such that $\mathcal{S} \cong \mathcal{A}/\mathcal{N}$ and $\mathcal{A} = \mathcal{S} \oplus \mathcal{N}$?

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This problem is an analogue to the validity of the Principal Wedderburn Theorem (PWT) for associative algebras.

Reduction preliminaries

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Let $\mathfrak{J}$ be a finite dimensional semisimple Jordan superalgebra, that is, $\mathcal{N}(\mathfrak{J}) = 0$ where $\mathcal{N}$ is the soluble radical. Fix a class $\mathfrak{M}(\mathfrak{J})$ of finite dimensional Jordan $\mathfrak{J}$-bimodules which is closed with respect to subbimodules and homomorphic images. Denote by $\mathfrak{K}_{\mathfrak{M}(\mathfrak{J})}$ the class of finite dimensional Jordan superalgebras $\mathcal{A}$ that satisfy the following conditions: $\mathcal{A}/\mathcal{N}(\mathcal{A}) \cong \mathfrak{J}$, $\mathcal{N}(\mathcal{A})^2 = 0$ and, $\mathcal{N}(\mathcal{A})$ considered as $\mathfrak{J}$-bimodule belongs to $\mathfrak{M}(\mathfrak{J})$. Then if PWT is true for all superalgebras $\mathcal{B} \in \mathfrak{K}_{\mathfrak{M}(\mathfrak{J})}$ with $\mathcal{N}(\mathcal{B})$ an irreducible $\mathfrak{J}$-bimodule, then it is true for all superalgebras $\mathcal{A}$ from $\mathfrak{K}_{\mathfrak{M}(\mathfrak{J})}$

irreducible bimodules over Jordan superalgebras

Irreducible bimodules over Jordan superalgebras of type $\mathcal{M}_{n|m}(\mathbb{F})^{(+)}$, $\mathcal{D}_t$, Kaplansky, and superform were classified by Zelmanov-Martinez. The cases of Jordan superalgebras of type K_{10} was proved by Shtern.

Answer to the problem

As a first step we consider the case in which the radical satifies $\mathcal{N}^2 = 0$, and the quotient superalgebra $\mathfrak{J}/\mathcal{N}$ is a simple Jordan superalgebra of one of the following types: $\mathcal{M}_{n|m}(\mathbb{F})^{(+)}$, superforms, $\mathcal{D}_t$, or $\mathcal{K}_{10}$, we prove that an analogue to the PWT is valid, provided some restrictions are imposed on the types of irreducible bimodules contained in the radical $\mathcal{N}$.

The restrictions are necessary and counter-examples were provided

Main Theorem

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Let $\mathfrak{J}, \mathcal{N}$ be as before. In the following cases there exists a subsuperalgebra $\mathcal{S} \subseteq \mathfrak{J}$ such that $\mathcal{S} \cong \mathfrak{J}/\mathcal{N}$ and $\mathfrak{J} = \mathcal{S} \oplus \mathcal{N}$:

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- *$\mathfrak{J}/\mathcal{N}$ is isomorphic to $\mathcal{M}_{n|m}(\mathbb{F})^{(+)}$. And when $n + m \geq 3$, $\mathcal{N}$ does not contain any copy of the regular bimodule $\operatorname{Reg} \mathcal{M}_{n|m}(\mathbb{F})^{(+)}$. When $m = n = 1$ $\mathcal{N}$ does not contain any copy of the regular bimodule nor of V^e.*

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- *$\mathfrak{J}/\mathcal{N}$ is a superalgebra of a superform with even part of dimension n, and $\mathcal{N}$ does not contain any copy of the irreducible bimodule $\mathcal{C}_n/\mathcal{C}_{n-2}$ when n is odd, or of $u \cdot \mathcal{C}_n / u \cdot \mathcal{C}_{n-2}$ when n is even.*

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- *$\mathfrak{J}/\mathcal{N}$ is isomorphic to $\mathcal{D}_t$, $t \neq -1$. And $\mathcal{N}$ does not contain any copy of the bimodule $\operatorname{Reg} \mathcal{D}_t$, or of the vector space generated by one even vector.*

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- *$\mathfrak{J}/\mathcal{N}$ is isomorphic to $\mathcal{D}_t$, $t \neq -1$. And $\mathcal{N}$ does not contain any copy of the bimodule $\operatorname{Reg} \mathcal{D}_t$, or of the vector space generated by one even vector.*
- *$\mathfrak{J}/\mathcal{N}$ is a Kac superalgebra without restriction in the bimodule.*

The restrictions impossed by the theorem above are essential, and we provide the corresponding counter-examples

Irreducible bimodules over Jordan superalgebras of superform

Let $V = V_0 + V_1$ be a vector superspace equipped with a nondegenerate superform, and suppose that $V_1 \neq 0$. Let $v_1, \ldots, v_n$ be an orthogonal basis of V_0 and $w_1, \ldots, w_{2m}$ be a basis of V_1 such that $(w_{2i-1}, w_{2i}) = 1$, $1 \leq i \leq m$, where all other products are zero.

Let $\mathcal{C}$ be the Clifford algebra over $\mathbb{F}$. Let $0 \leq i_1, \ldots, i_n \leq 1$ and $k_1, \ldots, k_{2m}$ are non negative integers, the elements $v_1^{i_1} \cdots v_n^{i_n} w_1^{k_1} \cdots w_{2m}^{k_{2m}}$, form a basis for $\mathcal{C}$.

Consider the subspace $\mathcal{C}_r = \sum_{i \leq r} \underbrace{V \cdots V}_{i}$ as the span of all basic products of lenght $\leq r$.

$$\mathbb{F} = \mathcal{C}_0 \subseteq \mathcal{C}_1 \subset \cdots ; \quad \mathcal{C} = \cup_{r \geq 0} \mathcal{C}_r$$

Irreducible bimodules over a Jordan superalgebras of superform

Any superspace of type $\mathcal{C}_r$ with r odd integer is a superbimodule over the Jordan superalgebra of superform.

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Theorem (C. Martinez- E. Zelmanov)

The only finite dimensional unital irreducible Jordan bimodules over $\mathfrak{J} = \mathbb{F} \cdot 1 + V$, (Jordan superalgebra of superform) are $\mathcal{C}_r/\mathcal{C}_{r-2}$ if r is odd and $u\mathcal{C}_r/u\mathcal{C}_{r-2}$ if r is even, where u is an even vector.

A counter-example to the PWT in Jordan superalgebras of superform

We consider the superspace $\mathcal{N} \subset \mathcal{A}$ where

$$\mathcal{A}_0 = \text{Vect}\left\langle 1, v_i, v_1 v_2, v_1 v_3, v_2 v_3, v_1 v_2 v_3, v_i w_s^2, v_i w_1 w_2, w_s^2, w_1 w_2 \right\rangle$$

for i=1,2,3, s=1,2

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$$\mathcal{A}_1 = \text{Vect}\,\langle\, w_1, w_2, v_i w_s, v_1 v_2 w_s, v_1 v_3 w_s, v_2 v_3 w_s, w_s^3, w_s^2 w_t \,\rangle$$

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$$\mathcal{N}_1 = \text{Vect}\langle v_i w_s, v_1 v_2 w_s, v_1 v_3 w_s, v_2 v_3 w_s, w_s^3, w_s^2 w_t \rangle$$

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\mathcal{A}_0 &= \text{Vect}\left\langle 1, v_i, v_1v_2, v_1v_3, v_2v_3, v_1v_2v_3, v_iw_s^2, v_iw_1w_2, w_s^2, w_1w_2 \right\rangle \\
\mathcal{A}_1 &= \text{Vect}\left\langle w_1, w_2, v_iw_s, v_1v_2w_s, v_1v_3w_s, v_2v_3w_s, w_s^3, w_s^2w_t \right\rangle \\
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\mathcal{N}_1 &= \text{Vect}\left\langle v_iw_s, v_1v_2w_s, v_1v_3w_s, v_2v_3w_s, w_s^3, w_s^2w_t \right\rangle
\end{aligned}$$

and we observe that

$$\mathcal{A}/\mathcal{N} \cong (\mathbb{F}\cdot 1 + \mathbb{F}\cdot v_1 + \mathbb{F}\cdot v_2 + \mathbb{F}\cdot v_3) \dot{+} (\mathbb{F}\cdot w_1 + \mathbb{F}\cdot w_2)$$

A counter-example

We define the nonzero products on $\mathcal{A}$, as follows $v_i^2 = 1$ for $i = 1, 2, 3$ and $w_1 \cdot w_2 = 1 + v_1 v_2 v_3$

$$v_i \circ v_j = \frac{1}{2}(v_i v_j + v_j v_i) \qquad 0 = v_i \circ w_s = v_i w_s + w_s v_i$$

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$$v_1 \cdot v_2 v_3 = v_1 v_2 v_3, \quad v_2 \cdot v_1 v_3 = -v_1 v_2 v_3, \quad v_3 \cdot v_1 v_2 = v_1 v_2 v_3,$$

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v_i \cdot w_1 w_2 = v_i w_1 w_2, & v_i \cdot w_s^2 = v_i w_s^2, & v_i \cdot v_i w_s^2 = w_s^2, \\
v_i \cdot v_i w_1 w_2 = w_1 w_2, & v_1 \cdot v_2 w_s = v_1 v_2 w_s, & v_1 \cdot v_3 w_s = v_1 v_3 w_s, \\
v_2 \cdot v_3 w_s = v_2 v_3 w_s, & v_i \cdot v_i v_j w_s = v_j w_s, &
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v_i \cdot w_1 w_2 = v_i w_1 w_2, & v_i \cdot w_s^2 = v_i w_s^2, & v_i \cdot v_i w_s^2 = w_s^2, \\
v_i \cdot v_i w_1 w_2 = w_1 w_2, & v_1 \cdot v_2 w_s = v_1 v_2 w_s, & v_1 \cdot v_3 w_s = v_1 v_3 w_s, \\
v_2 \cdot v_3 w_s = v_2 v_3 w_s, & v_i \cdot v_i v_j w_s = v_j w_s, & \\
w_s \cdot v_i v_j = v_i v_j w_s, & w_1 \cdot v_i v_j w_2 = v_i v_j, & w_2 \cdot v_i v_j w_1 = -v_i v_j,
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v_2 \cdot v_3 w_s = v_2 v_3 w_s, & v_i \cdot v_i v_j w_s = v_j w_s, & \\
w_s \cdot v_i v_j = v_i v_j w_s, & w_1 \cdot v_i v_j w_2 = v_i v_j, & w_2 \cdot v_i v_j w_1 = -v_i v_j, \\
w_1 \cdot w_2^3 = 3 w_2^2, & w_1 \cdot w_1 w_2^2 = 2 w_1 w_2, & w_1 \cdot w_1^2 w_2 = w_1^2 \\
w_2 \cdot w_1^3 = -3 w_1^2, & w_2 \cdot w_1 w_2^2 = -w_2^2, & w_2 \cdot w_1^2 w_2 = 2 w_1 w_2 \\
w_1 \cdot v_1 w_2^2 = -2 v_1 w_2, & w_1 \cdot v_1 w_1 w_2 = -v_1 w_1, & \\
w_2 \cdot v_1 w_1^2 = -2 v_1 w_1, & w_2 \cdot v_1 w_1 w_2 = v_1 w_2, &
\end{array}
$$

$$1 = w_1 \circ w_2 = \frac{1}{2}(w_1 w_2 - w_2 w_1)$$

A counter-example

it is easy to check that the superspace $\mathcal{A}$ with the multiplication above is a Jordan superalgebra and the quotient superalgebra is isomorphic to superalgebra of superform

$$\mathcal{A}/\mathcal{N} \quad \cong (\mathbb{F}\cdot 1 + \mathbb{F}\cdot v_1 + \mathbb{F}\cdot v_2 + \mathbb{F}\cdot v_3) \dot{+} (\mathbb{F}\cdot w_1 + \mathbb{F}\cdot w_2) \text{ e } \mathcal{N} \cong \mathcal{C}_3/\mathcal{C}_1.$$

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Since the PWT is valid for Jordan algebras, then there exist $\widetilde{v_i} \in \mathcal{A}_0$ such that $\widetilde{v_i} \equiv v_i \mod \mathcal{N}_0$ and $\widetilde{v_i}^2 = 1$, $\widetilde{v_i}\cdot\widetilde{v_j} = 0 \; i \neq j$.

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A counter-example

$(\mathcal{A}/\mathcal{N})_1 \cong \qquad \mathbb{F} \cdot w_1 + \mathbb{F} \cdot w_2$

If the PWT is valid for $\mathcal{A}$ then there exists elements $\widetilde{w_1}, \widetilde{w_2} \in \mathcal{A}_1$ such that $\widetilde{w_s} \equiv w_s$ and $\widetilde{w_1} \cdot \widetilde{w_2} = 1$, $\widetilde{w_s} \cdot v_i = 0$.

A counter-example

We recall that
$\mathcal{N}_1 = \text{Vect}\left\langle\, v_i w_s, v_1 v_2 w_s, v_1 v_3 w_s, v_2 v_3 w_s, w_s^3, w_s^2 w_t \right\rangle$, and we can adopt

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Now, it is possible to prove that

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therefore $\widetilde{w_1} \cdot \widetilde{w_2} = 1$, but $v_1 v_2 v_3$ is a non zero vector.

A counter-example

We recall that $w_1 \cdot w_2 = 1 + v_1 v_2 v_3$

$$\widetilde{w_1} = w_1 + \alpha_{11} w_1^3 + \alpha_{21} w_1^2 w_2 + \alpha_{31} w_1 w_2^2 + \alpha_{41} w_2^3$$
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therefore $\widetilde{w_1} \cdot \widetilde{w_2} = 1$ if and only if $v_1 v_2 v_3 = 0$, but $v_1 v_2 v_3$ is a non zero vector.

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THANKS FOR YOUR ATTENTION!