Peirce graded algebras of Jordan type

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Old dog...

Many people, including Felix and Sergey:

Motivated by Vertex Operator Algebras (VOA), consider algebras containing idempotents with eigenvalues

$$\left\{1, 0, \tfrac{1}{4}, \tfrac{1}{32}\right\}$$

Jon:

Consider algebras containing idempotents with eigenvalues

$$\{1, 0, \zeta, \eta\}$$

"Proof of concept" $\{1, 0, \eta\}$

New trick

Jon to Sergey and Felix:

`I (seem to) have a proof of the "Proof of concept" theorem. And it contains a surprise.`

Sergey to Jon:

`Thanks for the file. Indeed, this is a surprise. I was aware that the Jordan algebra case does not necessarily lead to a 3-transposition group, but this being the only exception is completely unexpected!`

Jon to Sergey:

`Jordan algebras?`

Algebras

For a vector space V over the field $\mathbb{F}$, an algebra structure is an $\mathbb{F}$-linear map multiplication:

$$V \otimes V \longrightarrow V \quad \text{given by} \quad v \otimes w \mapsto vw\,.$$

Alternatively, we have its adjoint actions—linear maps from V to its space of $\mathbb{F}$-endomorphisms:

$$v \mapsto \mathrm{LAd}_v \quad \text{with} \quad \mathrm{LAd}_v(w) = vw\,.$$

$$v \mapsto \mathrm{RAd}_v \quad \text{with} \quad \mathrm{RAd}_v(w) = wv\,.$$

The salient qualities of the algebra V are often given in terms of the representation theoretic behaviour of its adjoint image in the associative algebra $\mathrm{End}_{\mathbb{F}}(V)$.

Representation theory

A class of algebras is often characterized by properties of the adjoint action of single elements plus some type of Jacobi identity.

Examples

- Associative Algebra
- Lie Algebra
- Jordan Algebra

A Jordan algebra on V is given by the identities:

- (Single elements) $\mathrm{LAd}_v = \mathrm{RAd}_v = \mathrm{Ad}_v$.
- (Jacobi identity) $[\mathrm{Ad}_{v^2}, \mathrm{Ad}_v] = 0$.

E.g., $V^+ = (V, +, \circ)$ with $v \circ w = \frac{1}{2}(vw + wv)$.

(Associative $(V, +, \cdot)$ with $\mathrm{char}(\mathbb{F}) \neq 2$.)

Semisimplicity

- (Associative) The center is spanned by idempotents with Ad-eigenvalues $\{0, 1\}$.
- (Lie) There are a nondegenerate invariant form and Cartan elements and subalgebras $\mathfrak{sl}_2$ whose representation on V is highly restricted.
- (Jordan) There are a nondegenerate invariant form and idempotents with Ad-eigenvalues $\left\{0, 1, \frac{1}{2}\right\}$ and graded Ad-eigenspaces.

invariant form: $\langle\langle x, y\rangle\rangle = \langle\langle y, x\rangle\rangle$ and $\langle\langle xy, z\rangle\rangle = \langle\langle x, yz\rangle\rangle$

Vertex Operator Algebras

The VOA of interest here are given by, roughly:

- $V = \bigoplus_{n \geq 0} V_{(n)}$, a $\mathbb{C}$-space with $V_{(0)} = \mathbb{C}\mathbf{1},\ V_{(1)} = 0$.
- (Multiplication (many)) $V \otimes V \longrightarrow V[[z, z^{-1}]]$.
- (Single elements) various restrictions; e.g., $\mathbf{1} \otimes v \mapsto v\, z^0$.
- (Jacobi identity) For $v \in V$ let $\mathrm{Ad}_v(z)$ be the "adjoint power series" of endomorphisms of V. For $N = N(v, w) \gg 0$,

$$(z_1 - z_2)^N [\mathrm{Ad}_v(z_1), \mathrm{Ad}_w(z_2)] = 0 \,.$$

- (Semisimplicity I) V admits a positive definite form.
- (Semisimplicity II) There are conformal vectors $\omega \in V_{(2)}$ of central charge $\frac{1}{2}$ for which the coefficients of $\mathrm{Ad}_\omega(z)$ provide Virasoro subalgebras.

Automorphisms

The representation theory of Virasoro algebras led Miyamoto to the amazing result:

Miyamoto 1996

Each conformal vector of central charge $\frac{1}{2}$ gives a $\mathbb{Z}_2$-grading on these VOA that commutes with the original $\mathbb{N}$-grading. Thus these conformal vectors lead to a normal set of involutions in the automorphism group of the VOA respecting the grading.

For the Monster VOA $V^\natural$ associated with the Monster sporadic simple group $\mathbb{M}$, the Miyamoto involutions form the class of $2A$ (or "extra" or "triality") involutions.

Evolution

- (Griess 1981) constructs the Monster sporadic simple group $\mathbb{M}$ as the automorphism group of a nonassociative, commutative algebra in dimension 196884, the Griess algebra $\mathbb{G}$.
- (Frenkel, Lepowsky, Meurman 1988, Borcherds 1986) construct the Monster VOA $V^\natural$ with $\mathbb{G} = V^\natural_{(2)}$.
- (Miyamoto 1996) In every VOA V (of this type) the space $V_{(2)}$ yields a commutative (probably not associative) algebra admitting Miyamoto involutions—Griess algebras.
- (Ivanov 2007) isolates certain properties of the Griess algebras to define Majorana algebras.

Peirce grading

For m in the $\mathbb{F}$-algebra M and $\lambda \in \Lambda \subset \mathbb{F}$, let $M_\lambda(m)$ be the λ-eigenspace for Ad_m. (We allow $M_\lambda(m) = 0$.)

Majorana algebras are required to be generated by idempotents m (corresponding to conformal vectors) with Peirce decomposition

$$M = M_1(m) \oplus M_0(m) \oplus M_{\frac{1}{4}}(m) \oplus M_{\frac{1}{32}}(m)\,.$$

Furthermore $\Lambda_+ = \{1, 0, \frac{1}{4}\};\ \Lambda_- = \{\frac{1}{32}\}$ is a $\mathbb{Z}_2$-grading of M.

Therefore associated with m there is a Miyamoto involution $\tau(m)$ unless $M_{\frac{1}{32}} = 0$ in which case we instead take

$$\Lambda_+ = \{1, 0\}\,; \Lambda_- = \{\tfrac{1}{4}\}.$$

Axial algebras (JIH, FR, SS 2013)

Let disjoint Λ_+ and Λ_- be subsets of $\mathbb{F}$ (of characteristic not 2).

The commutative $\mathbb{F}$-algebra M is an axial algebra with respect to $\Lambda = \Lambda_+ \cup \Lambda_-$ if it is generated by a subset $\mathcal{P}$ of idempotent *axes* with:

- (Peirce decomposition) For $m \in \mathcal{P}$, $M = \bigoplus_{\lambda \in \Lambda} M_\lambda(m)$.
- ($\mathbb{Z}_2$-grading) For $m \in \mathcal{P}$ and $\delta, \epsilon \in \pm$

  $$M_\delta(m) M_\epsilon(m) \subseteq M_{\delta\epsilon}(m)\,,$$

  where $M_\star(m) = \bigoplus_{\lambda \in \Lambda_\star} M_\lambda(m)$.
- (Invariant form) $\langle\langle \cdot, \cdot \rangle\rangle$ is an invariant form from $M \times M$ to $\mathbb{F}$.
- (Primitivity) For $m \in \mathcal{P}$, $M_1(m) = \mathbb{F}m$ and $\langle\langle m, m \rangle\rangle = 1$.

Definitions and examples

Old dog:

The axial algebra has Virasoro type (ζ, η) provided
$$\Lambda_+ = \{1, 0, \zeta\} \text{ and } \Lambda_- = \{\eta\}$$

Example:
Majorana algebras are axial algebras of Virasoro type $\left(\frac{1}{4}, \frac{1}{32}\right)$

Proof of concept:

The axial algebra has Jordan type η provided
$$\Lambda_+ = \{1, 0\} \text{ and } \Lambda_- = \{\eta\}.$$

Examples: Majorana algebras for $M_\alpha = 0$ with $\{\alpha, \eta\} = \left\{\frac{1}{4}, \frac{1}{32}\right\}$;

surprise! : Jordan algebras for $\eta = \frac{1}{2}$.

A Sakuma theorem

The main working result:

"Proof of concept theorem" (JIH, FR, SS 2013)

Let M be an axial $\mathbb{F}$-algebra of Jordan type η generated by the two axes a_0 and a_1. Then we have one of:

- M is an associative algebra $\mathbb{F}$ and $|\tau(a_0)\tau(a_1)| = 1$.
- M is an associative algebra $\mathbb{F} \oplus \mathbb{F}$ and $|\tau(a_0)\tau(a_1)| = 2$.
- M is of type $3C(\eta)$ (or $3C(-1)^*$) and $|\tau(a_0)\tau(a_1)| = 3$.
- $\eta = \frac{1}{2}$ and M is a Jordan algebra of dimension at most 3.

Sakuma 2007 proved the corresponding 2-generator theorem for Griess algebras. (Adapted by Ivanov, Pasechnik, Seress, Shpectorov 2010 for Majorana algebras.)

Type 3C(η)

- $3C(\eta) = \mathbb{F}c_{-1} \oplus \mathbb{F}c_0 \oplus \mathbb{F}c_1$ subject to

$$c_i^2 = c_i\,,\ \ c_i c_j = \tfrac{\eta}{2}(c_i + c_j - c_k)$$

with symmetric bilinear form given by

$$\langle\langle c_i, c_i \rangle\rangle = 1\,,\ \ \langle\langle c_i, c_j \rangle\rangle = \tfrac{\eta}{2}\,.$$

- $3C(-1)^* = 3C(-1)/\mathbb{F}(c_{-1} + c_0 + c_1)$ and so is spanned by any two of d_{-1}, d_0, d_1 subject to

$$d_{-1} + d_0 + d_1 = 0\,,\ \ d_i^2 = d_i\,,\ \ d_i d_j = d_k$$

with

$$\langle\langle d_i, d_i \rangle\rangle = 1\,,\ \ \langle\langle d_i, d_j \rangle\rangle = -\tfrac{1}{2}\,.$$

Definition (Fischer 1971)

Let D be a conjugacy class of elements of order 2 in the group $G = \langle D \rangle$. Further assume that, for all $d, e \in D$, we have $|de| \leq 3$. Then (G, D) is called a 3-transposition group.

Motivating example: The tranpositions (i, j) of a symmetric group form a conjugacy class of 3-transpositions.

Corollary to the Sakuma theorem

Let M be a axial algebra of Jordan type $\eta \neq \frac{1}{2}$. Then the set of Miyamoto involutions corresponding to all η-axes form a normal set of 3-transpositions in the automorphism group of M, and M is a quotient of the $\mathbb{F}$-permutation module for this normal set.

This generalizes observations of Miyamoto ($\eta = \frac{1}{4}$) and Shpectorov ($\eta = \frac{1}{32}$).

This is significant because:

- Fischer's work (1971) together with that of Cuypers-JIH 1995 essentially classifies all 3-transposition groups.
  In particular, they are always locally finite. Hence a finitely generated axial algebra of Jordan type $\eta \neq \frac{1}{2}$ is finite dimensional.
- Results by Liebeck, Tiep, Sin, Hong, JIH, and others find almost all quotients of the corresponding permutation modules.
  In particular, the minimal eigenvalues of their Gram matrices are known (JIH, SS 2012), so those that are positive (semi-) definite can be identified.

Proving the Sakuma Theorem

(i) Set $\rho = \langle\langle a_0, a_1 \rangle\rangle / \eta$.
Also let $a_{-1} = a_1^{\tau(a_0)}$ and $a_2 = a_0^{\tau(a_1)}$ for the dihedral group $\langle \tau(a_0), \tau(a_1) \rangle$.

(ii) $M = \mathbb{F}a_0 + \mathbb{F}a_1 + \mathbb{F}a_2 = \mathbb{F}a_0 + \mathbb{F}a_1 + \mathbb{F}a_{-1}$.

(iii) $\langle\langle a_1 a_2, a_{-1} \rangle\rangle - \langle\langle a_1, a_2 a_{-1} \rangle\rangle = 2(\eta\rho - 1)\rho(2\rho - 1)(2\eta - 1)$.

As the form is invariant, this must be equal to 0.

The four ways this can happen give the four conclusions of the theorem.

A further piece of motivation

How easily can we get to

$\mathrm{P}\Omega_8^+(2){:}\,\mathsf{Sym}(3) \leq \mathrm{P}\Omega_8^+(3){:}\,\mathsf{Sym}(3) \leq \mathit{Fi}_{23}$

from

$\mathrm{P}\Omega_8^+(2){:}\,\mathsf{Sym}(3) \leq \mathrm{P}\Omega_8^+(3){:}\,\mathsf{Sym}(3)$??

Fischer 1973:

$\mathrm{P}\Omega_8^+(2){:}\,\mathsf{Sym}(3) \leq \mathrm{P}\Omega_8^+(3){:}\,\mathsf{Sym}(3) \leq \mathit{Fi}_{23} \leq \mathbb{BM} \leq \mathbb{M}$

Cuypers, Horn, in 't panhuis, Shpectorov 2012:

$\mathrm{P}\Omega_8^+(3){:}\,\mathsf{Sym}(3) \leq \mathit{Fi}_{23}$ simultaneously act on an

"$\mathbb{F}_2$-axial algebra with Jordan type $\eta = 2$ and dimension 782"

(although by our definitions, this makes no sense).